

Degenerate principal series of quantum Harish-Chandra modules

Olga Bershtein

Institute for Low Temperature Physics & Engineering
47 Lenin Ave., 61103 Kharkov, Ukraine
e-mail: bershtein@ilt.kharkov.ua
(Received)

Abstract

In this paper we study a quantum analogue of a degenerate principal series of $U_q\mathfrak{su}_{n,n}$ -modules ($0 < q < 1$) related to the Shilov boundary of the quantum $n \times n$ -matrix unit ball. We give necessary and sufficient conditions for the modules to be simple and unitarizable and investigate their equivalence.

These results are q -analogues of known classical results on reducibility and unitarizability of $SU(n, n)$ -modules obtained by Johnson, Sahi, Zhang, Howe and Tan.

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I Introduction

In this paper we investigate a quantum analogue of the degenerate principal series of representations of the algebra $U_q \mathfrak{su}_{n,n}$ related to the Shilov boundary of the quantum $n \times n$ -matrix unit ball. We give necessary and sufficient conditions for the representations to be irreducible and unitary.

In this work we provide q -analogues of classical results obtained by Kenneth D. Johnson, Siddhartha Sahi, Genkai Zhang, Roger E. Howe and Eng-Chye Tan [1, 2, 3, 4, 5]. Another degenerate principal series is considered in the A. Klimyk and S. Pakuliak paper [6].

We use Bargman's approach for investigating representations (see [7], where unitary strongly continuous irreducible representations of the group $SU(1, 1)$ were described). Explicit formulas for operators of $\mathfrak{su}_{1,1}$ -representations were found in a weight vectors basis in [7]. Results on irreducibility and unitarizability can be obtained from the formulas as corollaries.

In the general case one needs much more efforts to obtain similar formulas. Important results in this direction were obtained by Roger Howe in [1]. He received certain results on irreducibility and unitarizability of modules of the simplest degenerate principal series for $U(m, n)$ and some other classical groups.

The Lee Soo Teck paper [8] directly continues this Howe work. In [8] the degenerate principal series for $U(n, n)$ related to the Shilov boundary of the $n \times n$ -quantum ball is investigated and answers to the same questions are obtained.

This work generalizes results from [8] to the quantum case with $0 < q < 1$. Passing to the limit as $q \mapsto 1$ one can get up to notation the results of the above-mentioned paper.

This paper is organized as follows. In Section II we define the representations $\pi_{\alpha,\beta}$ of the degenerate principal series (see (5)). In Section III we investigate the equivalence of $\pi_{\alpha,\beta}$ (see Proposition 3). In Section IV we discuss some auxiliary results $\pi_{\alpha,\beta}$. These results will be used in the sequel. In Section V we give necessary and sufficient for $\pi_{\alpha,\beta}$ to be irreducible (see Proposition 11). For the case $\pi_{\alpha,\beta}$ is reducible, we describe all its irreducible subrepresentations. In Section VI we find explicit formulas for intertwining operators between $\pi_{\alpha,\beta}$ and $\pi_{\alpha,\beta}$ (see (17)). In Section VII we investigate unitarizability of irreducible representations of the degenerate principal series. Most of the technical details of the proofs are contained in Appendix.

II Definition of the degenerate principal series of representations

Recall some concepts on geometric realizations for certain series of representations of real semisimple Lie groups and Lie algebras.

Consider the affine algebraic group $G = SL_{2n}(\mathbb{C})$ and its maximal parabolic subgroup

$$P = \left\{ \begin{pmatrix} A & B \\ 0 & D \end{pmatrix} \mid A, B, D \in Mat_{n,n}(\mathbb{C}), (\det A)(\det D) = 1 \right\}.$$

Then the projective variety G/P is isomorphic to the space $Gr_n(\mathbb{C}^{2n})$ of n -dimensional subspaces in \mathbb{C}^{2n} . The subgroup $K = S(GL_n(\mathbb{C}) \times GL_n(\mathbb{C}))$ acts naturally on G/P .

Denote by Ω the open K -orbit. It can be easily proved that

$$\Omega = \{L \in Gr_n(\mathbb{C}^{2n}) \mid \dim L \cap (\mathbb{C}^n)_1 = \dim L \cap (\mathbb{C}^n)_2 = 0\},$$

where $(\mathbb{C}^n)_1$ and $(\mathbb{C}^n)_2$ are the subspaces generated by the elements $\{\varepsilon_1, \dots, \varepsilon_n\}$, $\{\varepsilon_{n+1}, \dots, \varepsilon_{2n}\}$ of the standard basis for \mathbb{C}^{2n} , respectively. It can be verified that Ω is an affine variety.

Set

$$\mathbf{t} = \begin{pmatrix} t_{11} & t_{12} & \dots & t_{12n} \\ \dots & \dots & \dots & \dots \\ t_{n1} & t_{n2} & \dots & t_{n2n} \end{pmatrix}, \quad \text{rk } \mathbf{t} = n,$$

and $\mathbb{C}[\text{Mat}_{n,2n}] \stackrel{\text{def}}{=} \mathbb{C}[t_{11}, \dots, t_{n2n}]$. Define

$$t_J^{\wedge n} \stackrel{\text{def}}{=} \sum_{s \in S_n} (-1)^{l(s)} t_{1j_{s(1)}} \cdot t_{2j_{s(2)}} \cdots t_{nj_{s(n)}},$$

where $l(s)$ is the length of permutation s , $J = \{j_1, \dots, j_n\}$, $1 \leq j_1 < \dots < j_n \leq 2n$, and t_{ij} are the matrix entries of \mathbf{t} . The elements $t_J^{\wedge n}$ are called Plucker projective "coordinates" on $Gr_n(\mathbb{C}^{2n})$. Denote by $\mathbb{C}[\text{Pl}_{n,2n}] \subset \mathbb{C}[\text{Mat}_{n,2n}]$ the subalgebra generated by all $t_J^{\wedge n}$.

Consider the algebra $\mathbb{C}[\Omega]$ of regular functions on Ω . Let us introduce some notation. Set $t \stackrel{\text{def}}{=} t_{\{n+1, \dots, 2n\}}^{\wedge n}$ and

$$z_a^b = t^{-1} t_{J_{ab}}^{\wedge n}, \quad a, b = 1, \dots, n, \quad \text{where } J_{ab} = \{n+1, \dots, 2n\} \setminus \{2n+1-b\} \cup \{a\};$$

$$\mathbf{z} = \begin{pmatrix} z_1^1 & \dots & z_1^n \\ \dots & \dots & \dots \\ z_n^1 & \dots & z_n^n \end{pmatrix}, \quad \det \mathbf{z} = \det \begin{pmatrix} z_1^1 & \dots & z_1^n \\ \dots & \dots & \dots \\ z_n^1 & \dots & z_n^n \end{pmatrix}.$$

Then the algebra $\mathbb{C}[\Omega]$ is canonically isomorphic to the localization of the algebra $\mathbb{C}[\text{Mat}_n] \stackrel{\text{def}}{=} \mathbb{C}[z_1^1, \dots, z_n^n]$ with respect to the multiplicative set $(\det \mathbf{z})^{\mathbb{Z}_+}$. The vector space $\mathbb{C}[\Omega] = \mathbb{C}[\text{Mat}_n]_{\det \mathbf{z}}$ can be naturally equipped with an \mathfrak{sl}_{2n} -module structure and a K -module structure, and these structures are compatible (see [9]).

Therefore the action of the universal enveloping algebra $U\mathfrak{sl}_{2n}$ in the vector space $\mathbb{C}[\text{Mat}_n]_{\det \mathbf{z}}$ is well defined. Moreover, the $U\mathfrak{sl}_{2n}$ -action in the localization of the algebra $\mathbb{C}[\text{Pl}_{n,2n}]$ with respect to the multiplicative set $t^{\mathbb{Z}_+}$ is well defined. Hence the $U\mathfrak{sl}_{2n}$ -action in the space¹ $\mathbb{C}[\text{Mat}_n]_{\det \mathbf{z}} \cdot (\det \mathbf{z})^{\alpha} t^{\beta}$ is well defined for each $\alpha, \beta \in \mathbb{Z}$.

Now let us pass to the quantum case. Everywhere in the sequel $q \in (0, 1)$, \mathbb{C} is the ground field and all algebras are unital.

Denote by $U_q \mathfrak{sl}_{2n}$ the algebra defined by its generators $\{E_i, F_i, K_i, K_i^{-1}\}_{i=1}^{2n-1}$ and the relations

$$\begin{aligned} K_i K_j &= K_j K_i, \quad K_i K_i^{-1} = K_i^{-1} K_i = 1; \\ K_i E_i &= q^2 E_i K_i, \quad K_i F_i = q^{-2} F_i K_i; \\ E_i F_j - F_j E_i &= \delta_{ij} (K_i - K_i^{-1}) / (q - q^{-1}); \\ K_i E_j &= q^{-1} E_j K_i, \quad K_i F_j = q F_j K_i, \quad |i - j| = 1; \end{aligned}$$

¹They are spaces of sections of homogeneous vector bundles over Ω . We pass from $\alpha, \beta \in \mathbb{Z}$ to $\alpha, \beta \in \mathbb{R}$ standardly.

$$E_i^2 E_j - (q + q^{-1}) E_i E_j E_i + E_j E_i^2 = 0, \quad |i - j| = 1;$$

$$F_i^2 F_j - (q + q^{-1}) F_i F_j F_i + F_j F_i^2 = 0, \quad |i - j| = 1;$$

$$K_i E_j - E_j K_i = K_i F_j - F_j K_i, = E_i E_j - E_j E_i = F_i F_j - F_j F_i = 0, \quad |i - j| > 1.$$

We equip $U_q \mathfrak{sl}_{2n}$ with the standard Hopf algebra structure. The comultiplication, the counit and the antipode are defined by their actions on the generators:

$$\begin{aligned} \triangle E_j &= E_j \otimes 1 + K_j \otimes E_j, & \varepsilon(E_j) &= 0, & S(E_j) &= -K_j^{-1} E_j, \\ \triangle F_j &= F_j \otimes K_j^{-1} + 1 \otimes F_j, & \varepsilon(F_j) &= 0, & S(F_j) &= -F_j K_j, \\ \triangle K_j &= K_j \otimes K_j, & \varepsilon(K_j) &= 1, & S(K_j) &= K_j^{-1} \end{aligned}$$

for all $j = 1, \dots, 2n - 1$.

The algebra $\mathbb{C}[\text{Mat}_{n,2n}]_q$ of polynomials on the quantum $n \times 2n$ -matrix space is defined by its generators $\{t_{ij}\}_{i=1,\dots,n;j=1,\dots,2n}$ and the relations (cf. [10])

$$\begin{aligned} t_{ik} t_{jk} &= q t_{jk} t_{ik}, & t_{ki} t_{kj} &= q t_{kj} t_{ki}, & i < j, \\ t_{ij} t_{kl} &= t_{kl} t_{ij}, & i < k \text{ \& } j > l, \\ t_{ij} t_{kl} - t_{kl} t_{ij} &= (q - q^{-1}) t_{ik} t_{jl}, & i < k \text{ \& } j < l. \end{aligned} \quad (1)$$

Define q-minors as follows:

$$t_{IJ}^{\wedge k} \stackrel{\text{def}}{=} \sum_{s \in S_k} (-q)^{l(s)} t_{i_1 j_{s(1)}} \cdots t_{i_k j_{s(k)}}, \quad (2)$$

for any $I = \{i_1, \dots, i_k\}$, $1 \leq i_1 < \dots < i_k \leq n$, $J = \{j_1, \dots, j_k\}$, $1 \leq j_1 < \dots < j_k \leq 2n$; here $l(s)$ denotes the length of permutation s .

Consider the algebra $\mathbb{C}[\text{Pl}_{n,2n}]_q \subset \mathbb{C}[\text{Mat}_{n,2n}]_q$ generated by all q-minors $t_{\{1,\dots,n\}J}^{\wedge n}$, card $J = n$. It is equipped with the standard $U_q \mathfrak{sl}_n^{\text{op}} \otimes U_q \mathfrak{sl}_{2n}$ -module algebra structure.² It is easy to show that the $U_q \mathfrak{sl}_n^{\text{op}}$ -structure can be reconstructed from the below equalities:

$$\begin{aligned} K_l t_{ij} &= \begin{cases} q^{-1} t_{ij}, & l = i, \\ q t_{ij}, & l = i - 1, \\ 0, & \text{otherwise;} \end{cases} \\ E_l t_{ij} &= q^{-1/2} \cdot \begin{cases} t_{(i+1)j}, & l = i, \\ 0, & \text{otherwise;} \end{cases} \quad F_l t_{ij} = q^{1/2} \cdot \begin{cases} t_{(i-1)j}, & l = i - 1, \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

The element $t \stackrel{\text{def}}{=} t_{\{1,2,\dots,n\}\{n+1,n+2,\dots,2n\}}^{\wedge n}$ quasi-commutes with t_{ij} for all $i = 1, \dots, n$, $j = 1, \dots, 2n$ and is $U_q \mathfrak{sl}_n^{\text{op}}$ -invariant.

Denote by $\mathbb{C}[\text{Pl}_{n,2n}]_{q,t}$ the localization of the algebra $\mathbb{C}[\text{Pl}_{n,2n}]_q$ with respect to the multiplicative system $t^{\mathbb{Z}^+}$. Introduce q-analogues of coordinates on Ω as follows:

$$z_a^b \stackrel{\text{def}}{=} t^{-1} t_{\{1,2,\dots,n\}J_{ab}}^{\wedge n}, \quad (3)$$

where $J_{ab} = \{n+1, n+2, \dots, 2n\} \setminus \{2n+1-b\} \cup \{a\}$.

² $U_q \mathfrak{sl}_n^{\text{op}}$ is a Hopf algebra with the same multiplication and the opposite comultiplication.

The defining relations for the subalgebra generated by the elements z_a^b are obtained in [11]:

$$\begin{aligned} z_a^{b_1} z_a^{b_2} &= q z_a^{b_2} z_a^{b_1}, & b_1 < b_2, \\ z_{a_1}^b z_{a_2}^b &= q z_{a_2}^b z_{a_1}^b, & a_1 < a_2, \\ z_{a_1}^{b_1} z_{a_2}^{b_2} &= z_{a_2}^{b_2} z_{a_1}^{b_1}, & b_1 < b_2 \text{ \& } a_1 > a_2, \\ z_{a_1}^{b_1} z_{a_2}^{b_2} - z_{a_2}^{b_2} z_{a_1}^{b_1} &= (q - q^{-1}) z_{a_1}^{b_2} z_{a_2}^{b_1}, & b_1 < b_2 \text{ \& } a_1 < a_2. \end{aligned}$$

(For the special case $n = 2$ see the Noumi paper [12].)

It can be checked easily that $zt = qtz$ for any $z \in \{z_a^b | a, b = 1, \dots, n\}$.

It can be proved that for any $\xi \in U_q \mathfrak{sl}_{2n}$, $f \in \mathbb{C}[\text{Pl}_{n,2n}]_{q,t}$ there is a unique Laurent polynomial $p_{f,\xi}$ of the variable $u = q^k$ with coefficients in $\mathbb{C}[\text{Pl}_{n,2n}]_{q,t}$ such that $p_{f,\xi}(q^k) = \xi \cdot (ft^k)t^{-k}$. This allows one to prove the existence of an extension of $U_q \mathfrak{sl}_{2n}$ -module algebra structure onto $\mathbb{C}[\text{Pl}_{n,2n}]_{q,t}$ (see [13]).

The subalgebra generated by z_a^b is the algebra $\mathbb{C}[\text{Mat}_n]_q$ of "polynomials on the quantum $n \times n$ -matrix space" (cf. (1)). The algebra $\mathbb{C}[\text{Mat}_n]_q$ is a $U_q \mathfrak{sl}_{2n}$ -module subalgebra of the $U_q \mathfrak{sl}_{2n}$ -module algebra $\mathbb{C}[\text{Pl}_{n,2n}]_{q,t}$ (see [14]).

Proposition 1 ([14]) *For all $a, b = 1, \dots, n$*

$$K_n^{\pm 1} z_a^b = \begin{cases} q^{\pm 2} z_a^b, & a = n \text{ \& } b = n \\ q^{\pm 1} z_a^b, & a = n \text{ \& } b \neq n \text{ or } a \neq n \text{ \& } b = n \\ z_a^b, & \text{otherwise,} \end{cases}$$

$$F_n z_a^b = q^{1/2} \cdot \begin{cases} 1, & a = n \text{ \& } b = n \\ 0, & \text{otherwise,} \end{cases} \quad E_n z_a^b = -q^{1/2} \cdot \begin{cases} q^{-1} z_a^n z_n^b, & a \neq n \text{ \& } b \neq n \\ (z_n^n)^2, & a = n \text{ \& } b = n \\ z_n^n z_a^b, & \text{otherwise} \end{cases}$$

and for all $k \neq n$ we have

$$\begin{aligned} K_k^{\pm 1} z_a^b &= \begin{cases} q^{\pm 1} z_a^b, & k < n \text{ \& } a = k \text{ or } k > n \text{ \& } b = 2n - k, \\ q^{\mp 1} z_a^b, & k < n \text{ \& } a = k + 1 \text{ or } k > n \text{ \& } b = 2n - k + 1, \\ z_a^b, & \text{otherwise,} \end{cases} \\ F_k z_a^b &= q^{1/2} \cdot \begin{cases} z_{a+1}^b, & k < n \text{ \& } a = k, \\ z_a^{b+1}, & k > n \text{ \& } b = 2n - k, \\ 0, & \text{otherwise,} \end{cases} \quad E_k z_a^b = q^{-1/2} \cdot \begin{cases} z_{a-1}^b, & k < n \text{ \& } a = k + 1, \\ z_a^{b-1}, & k > n \text{ \& } b = 2n - k + 1, \\ 0, & \text{otherwise.} \end{cases} \quad \square \end{aligned}$$

In the sequel we use the following notation for q-minors

$$\mathbf{z}^{\wedge k \{b_1, \dots, b_k\}}_{\{a_1, \dots, a_k\}} \stackrel{\text{def}}{=} \sum_{s \in S_k} (-q)^{l(s)} z_{a_1}^{b_{s(1)}} \dots z_{a_k}^{b_{s(k)}}, \quad (4)$$

where $a_1 < \dots < a_k$, $b_1 < \dots < b_k$. It is known that the element $\det_q \mathbf{z} \stackrel{\text{def}}{=} \mathbf{z}^{\wedge n \{1, \dots, n\}}_{\{1, \dots, n\}}$ belongs to the center of $\mathbb{C}[\text{Mat}_n]_q$ and $\mathbb{C}[\text{Mat}_n]_q$ has no zero divisors.

Denote by $\mathbb{C}[\text{Mat}_n]_{q, \det_q \mathbf{z}}$ the localization of the algebra $\mathbb{C}[\text{Mat}_n]_q$ with respect to the multiplicative system $(\det_q \mathbf{z})^{\mathbb{Z}^+}$. We consider $\mathbb{C}[\text{Mat}_n]_{q, \det_q \mathbf{z}}$ as a q-analogue of the space

of regular functions on the open orbit Ω . Let $\tilde{t} = t^{\wedge_{\{1,\dots,n\}}^n}_{\{1,\dots,n\}}$. Since $\det_q \mathbf{z} = t^{-1}\tilde{t}$, we see that the algebra $\mathbb{C}[\text{Mat}_n]_{q,\det_q \mathbf{z}}$ is a $U_q \mathfrak{sl}_{2n}$ -module subalgebra of the $U_q \mathfrak{sl}_{2n}$ -module algebra $\mathbb{C}[\text{Pl}_{n,2n}]_{q,t,\tilde{t}}$. (As above, to verify that the extension is well defined we use the following fact: for all $\xi \in U_q \mathfrak{sl}_{2n}$, $f \in V$ the vector valued function $\xi \cdot (f(\det_q \mathbf{z})^k)(\det_q \mathbf{z})^{-k}$ is a Laurent polynomial of the variable $u = q^k$.)

Denote by V the vector space $\mathbb{C}[\text{Mat}_n]_{q,\det_q \mathbf{z}}$. Assume first that $\alpha, \beta \in \mathbb{Z}$. Define a representation $\pi_{\alpha,\beta} : U_q \mathfrak{sl}_{2n} \rightarrow \text{End} V$ as follows:

$$\pi_{\alpha,\beta}(\xi)f = (\xi \cdot (f(\tilde{t})^\alpha t^\beta))t^{-\beta}(\tilde{t})^{-\alpha} = (\xi \cdot (f(\det_q \mathbf{z})^\alpha t^{\beta+\alpha}))t^{-\alpha-\beta}(\det_q \mathbf{z})^{-\alpha} \quad (5)$$

for every $\xi \in U_q \mathfrak{sl}_{2n}$, $f \in V$. For each $\lambda \in \mathbb{Z}$ we have

$$E_j t^\lambda = 0, \quad F_j t^\lambda = 0, \quad K_j t^\lambda = t^\lambda, \quad j = 1, \dots, 2n-1, \quad j \neq n$$

$$E_n t^\lambda = q^{-3/2} \frac{1 - q^{-2\lambda}}{1 - q^{-2}} z_n^n t^\lambda, \quad F_n t^\lambda = 0, \quad K_n^\pm t^\lambda = q^{\mp \lambda} t^\lambda,$$

$$E_j (\det_q \mathbf{z})^\lambda = 0, \quad F_j (\det_q \mathbf{z})^\lambda = 0, \quad K_j (\det_q \mathbf{z})^\lambda = (\det_q \mathbf{z})^\lambda, \quad j = 1, \dots, 2n-1, \quad j \neq n$$

$$K_n^\pm ((\det_q \mathbf{z})^\lambda) = q^{\pm 2\lambda} (\det_q \mathbf{z})^\lambda, \quad E_n ((\det_q \mathbf{z})^\lambda) = -q^{1/2} \frac{1 - q^{2\lambda}}{1 - q^2} z_n^n (\det_q \mathbf{z})^\lambda,$$

$$F_n ((\det_q \mathbf{z})^\lambda) = q^{1/2} \frac{1 - q^{-2\lambda}}{1 - q^{-2}} z_{\{1,\dots,n-1\}}^{\wedge_{\{1,\dots,n-1\}}^{n-1}} (\det_q \mathbf{z})^{\lambda-1}, \quad \lambda \neq 0.$$

From these equalities we see that for each $\xi \in U_q \mathfrak{sl}_{2n}$, $f \in V$ the vector valued function $p_{f,\xi}(q^\alpha, q^\beta) \stackrel{\text{def}}{=} \pi_{\alpha,\beta}(\xi)(f)$ is a Laurent polynomial of the variables q^α, q^β . These Laurent polynomials are defined by their values on the set $\{(q^\alpha, q^\beta) \mid \alpha, \beta \in \mathbb{Z}\}$ and deliver the canonical "analytic continuation" for $\pi_{\alpha,\beta}(\xi)(f)$ to $(\alpha, \beta) \in \mathbb{C}^2$.

Let $(\alpha, \beta) \in \mathbb{C}^2$. Define a representation $\pi_{\alpha,\beta}(\xi)(f) \stackrel{\text{def}}{=} p_{f,\xi}(q^\alpha, q^\beta)$. Indeed, to prove that the representation $\pi_{\alpha,\beta}$ is well defined for $(q^\alpha, q^\beta) \in \mathbb{C}^2$ it is sufficient to verify some identities for Laurent polynomials. These identities are correct for $\alpha, \beta \in \mathbb{Z}$.

Introduce a "deformation parameter" h by the equality $q = e^{-h/2}$. Clearly, if $\alpha_1 = \alpha_2 + i\frac{2\pi}{h}$ and $\beta_1 = \beta_2 + i\frac{2\pi}{h}$, then $\pi_{\alpha_1,\beta_1} = \pi_{\alpha_2,\beta_2}$. Then it is enough to consider $\alpha, \beta \in D$, where

$$D = \{\alpha, \beta \in \mathbb{C} \mid 0 \leq \text{Im } \alpha < \frac{2\pi}{h}, 0 \leq \text{Im } \beta < \frac{2\pi}{h}\}.$$

Recall that a representation $\rho : U_q \mathfrak{sl}_{2n} \rightarrow \text{End} W$ is called *weight* if the representation space W decomposes as follows:

$$W = \bigoplus_{\lambda} W_{\lambda}, \quad \text{where } \lambda = (\lambda_1, \dots, \lambda_{2n-1}) \in \mathbb{Z}^{2n-1},$$

$$W_{\lambda} = \{v \in W \mid \rho(K_j^\pm)v = q^{\pm \lambda_j}v, j = 1, \dots, 2n-1\}.$$

The subspace W_{λ} is called weight subspace with weight λ . In the sequel we will consider only weight representations. It is clear that $\pi_{\alpha,\beta}$ is a weight representation if and only if $q^{\alpha-\beta} \in q^{\mathbb{Z}}$.

Let W be a weight $U_q \mathfrak{sl}_{2n}$ -module. Define operators H_i for $i = 1, \dots, 2n-1$ by the formula $H_i|_{W_{\lambda}} = \lambda_i$.

III Equivalence of the representations

Recall that $q = e^{-h/2}$. For any complex α, β such that $0 \leq \text{Im } \alpha < \frac{2\pi}{h}$, $0 \leq \text{Im } \beta < \frac{2\pi}{h}$, the statements $\alpha - \beta \in \mathbb{Z}$ and $q^{\alpha-\beta} \in q^{\mathbb{Z}}$ are equivalent.

Proposition 2 *If $\alpha, \beta \notin \mathbb{Z}$, then the representations $\pi_{\alpha, \beta}$ and $\pi_{-n-\beta, -n-\alpha}$ are equivalent.*

The proof is reduced to explicit formulas for the intertwining operators. It is given in Section VI.

If $\alpha, \beta \in \mathbb{Z}$, then the representations $\pi_{\alpha, \beta}$ and $\pi_{-n-\beta, -n-\alpha}$ are not equivalent. This fact follows from the statement that only one of the representations $\pi_{\alpha, \beta}$ and $\pi_{-n-\beta, -n-\alpha}$ for integral α, β has a finite dimensional subrepresentation. An explanation of this fact is given in the end of Section V.

The representations $\pi_{\alpha, \beta}$ and $\pi_{\alpha-1, \beta+1}$ are equivalent for all α, β . The corresponding intertwining operator $T : V \rightarrow V$ is defined as follows: for every $f \in V = \mathbb{C}[\text{Mat}_n]_{q, \det_q \mathbf{z}}$ $T(f) = f(\det_q \mathbf{z})^{-1}$. Indeed, since for each $f \in V$, $\xi \in U_q \mathfrak{sl}_{2n}$

$$\begin{aligned} \pi_{\alpha-1, \beta+1}(\xi)(f) &= (\xi \cdot (f(\det_q \mathbf{z})^{\alpha-1} t^{\beta+\alpha})) t^{-\alpha-\beta} (\det_q \mathbf{z})^{1-\alpha} = \\ &= (\xi \cdot (f(\det_q \mathbf{z})^{-1} (\det_q \mathbf{z})^{\alpha} t^{\beta+\alpha})) t^{-\alpha-\beta} (\det_q \mathbf{z})^{-\alpha} (\det_q \mathbf{z}) = \pi_{\alpha, \beta}(\xi)(f(\det_q \mathbf{z})^{-1}) \det_q \mathbf{z}, \end{aligned}$$

we see that T intertwines the representations $\pi_{\alpha, \beta}$ and $\pi_{\alpha-1, \beta+1}$. Therefore without loss of generality we can assume that $\alpha, \beta \in \mathcal{D}$, where

$$\mathcal{D} = \{(\alpha, \beta) \in \mathbb{C} \mid \alpha - \beta \in \{0, 1\}, 0 \leq \text{Im } \alpha < \frac{2\pi}{h}, 0 \leq \text{Im } \beta < \frac{2\pi}{h}\}. \quad (6)$$

Let us introduce an equivalence relation on \mathcal{D} . The equivalence class of (α, β) consists of one point for $\alpha, \beta \in \mathbb{Z}$ and from two points for $\alpha, \beta \notin \mathbb{Z}$:

$$(\alpha_1, \beta_1) \sim (\alpha_2, \beta_2), \quad \text{iff} \quad \begin{cases} \alpha_1 = -n - \beta_2, \beta_1 = -n - \alpha_2 \text{ for } \text{Im } \alpha_1 = \text{Im } \alpha_2 = 0, \\ \alpha_1 = \frac{2\pi i}{h} - n - \beta_2, \beta_1 = \frac{2\pi i}{h} - n - \alpha_2, \text{ otherwise.} \end{cases}$$

Proposition 3 *The set of equivalence classes \mathcal{D}/\sim is in the one-to-one correspondence $(\alpha, \beta) \mapsto \pi_{\alpha, \beta}$ with the set of equivalence classes of the representations of the degenerate principal series.*

Proof. By the above, each representation of the degenerate principal series is equivalent to the representation $\pi_{\alpha, \beta}$ for some $(\alpha, \beta) \in \mathcal{D}$.

Prove that the representations π_{α_1, β_1} and π_{α_2, β_2} , with $(\alpha_1, \beta_1), (\alpha_2, \beta_2) \in \mathcal{D}$, are equivalent if and only if $(\alpha_1, \beta_1) \sim (\alpha_2, \beta_2)$. For that we calculate the action of a central element $C \in U_q \mathfrak{sl}_{2n}^{\text{ext}}$ (see [15] for the definition). It can be proved that $\pi_{\alpha, \beta}(C)$ is a scalar operator for all $\alpha, \beta \in \mathcal{D}$.

From [16] it follows that there exists a unique central element C which acts on the $U_q \mathfrak{sl}_{2n}$ -highest vector v^{high} with weight λ as follows:

$$C(v^{\text{high}}) = \sum_{j=0}^{2n-1} q^{-2(\mu_j, \lambda + \rho)} v^{\text{high}},$$

where $\mu_0 = \varpi_1$, $\mu_j = -\varpi_j + \varpi_{j+1}$ for $j = 1, \dots, 2n-2$, $\mu_{2n-1} = -\varpi_{2n-1}$, ϖ_j are the fundamental weights, 2ρ is the sum of positive roots of the Lie algebra \mathfrak{sl}_{2n} , and we choose the invariant scalar product such that $(\alpha, \alpha) = 2$ for any simple root α .

First let α, β be integers. It can be proved that

$$\pi_{\alpha, \beta}(C)(\det_q \mathbf{z})^\beta = 4 \operatorname{ch} \frac{h}{2}(\alpha + \beta + n) \left(\sum_{j=0}^{n-1} \operatorname{ch} \frac{h}{2} j \right) (\det_q \mathbf{z})^\beta.$$

Hence $\pi_{\alpha, \beta}(C) = 4 \operatorname{ch} \frac{h}{2}(\alpha + \beta + n) \left(\sum_{j=0}^{n-1} \operatorname{ch} \frac{h}{2} j \right) \cdot \operatorname{Id}$ for all $(\alpha, \beta) \in \mathcal{D}$.

Suppose that π_{α_1, β_1} and π_{α_2, β_2} are equivalent. Equivalent representations have the same weight lattice. Therefore $(\alpha_1 - \beta_1) - (\alpha_2 - \beta_2) \in 2\mathbb{Z}$. Since $(\alpha_1, \beta_1), (\alpha_2, \beta_2) \in \mathcal{D}$, we see that $(\alpha_1 - \beta_1) - (\alpha_2 - \beta_2) = 0$.

Then the equivalent representations π_{α_1, β_1} and π_{α_2, β_2} have the same values of central characters, which means that

$$\left(\operatorname{ch} \frac{h}{2}(\alpha_1 + \beta_1 + n) - \operatorname{ch} \frac{h}{2}(\alpha_2 + \beta_2 + n) \right) \sum_{j=0}^{n-1} \operatorname{ch} \frac{h}{2} j = 0$$

Since $0 \leq \operatorname{Im} \alpha_1 < \frac{2\pi}{h}$, $0 \leq \operatorname{Im} \beta_1 < \frac{2\pi}{h}$, $0 \leq \operatorname{Im} \alpha_2 < \frac{2\pi}{h}$, $0 \leq \operatorname{Im} \beta_2 < \frac{2\pi}{h}$, we have that $\alpha_1 + \beta_1 = \alpha_2 + \beta_2$, or $\alpha_1 + \beta_1 = -\alpha_2 - \beta_2 - 2n$, or $\alpha_1 + \beta_1 = -\alpha_2 - \beta_2 - 2n - \frac{4\pi i}{h}$. If $\alpha_1 + \beta_1 = \alpha_2 + \beta_2$, then $\alpha_1 = \alpha_2$ and $\beta_1 = \beta_2$. For any fixed non-integral α_1, β_1 there is a unique pair $(\alpha_2, \beta_2) \in \mathcal{D}$ such that either $\alpha_1 + \beta_1 = -\alpha_2 - \beta_2 - 2n$ or $\alpha_1 + \beta_1 = -\alpha_2 - \beta_2 - 2n - \frac{4\pi i}{h}$, and $(\alpha_1, \beta_1) \sim (\alpha_2, \beta_2)$. Although for integral parameters π_{α_1, β_1} and π_{α_2, β_2} are not equivalent, because the only one of them has a finite-dimensional subrepresentation. This can be deduced from Corollary 4. Thus each equivalence class in \mathcal{D} is assigned to a unique equivalence class of the representations of the degenerate principal series $\pi_{\alpha, \beta}$. \square

IV Auxiliary statements on $\pi_{\alpha, \beta}$ -structure

In this section we describe some necessary technical results, that will be useful in the sequel.

Everywhere in this section we assume that $n > 1$. However, Propositions 4, 7, and 10 and Corollaries 1 and 2 are still sensible and correct for $n = 1$.

Let $U_q \mathfrak{k}_{ss} \subset U_q \mathfrak{sl}_{2n}$ be the Hopf subalgebra generated by $E_j, F_j, K_j^{\pm 1}$, $j = 1, \dots, 2n-1$, $j \neq n$ and $U_q \mathfrak{k} \subset U_q \mathfrak{sl}_{2n}$ be the Hopf subalgebra generated by $K_n^{\pm 1}$ and $U_q \mathfrak{k}_{ss}$.

Note that $\pi_{\alpha, \beta}|_{U_q \mathfrak{k}_{ss}}$ does not depend on α, β . The following preliminary result on reducibility of $\pi_{\alpha, \beta}$ is well known in the classical case. For brevity, set³

$$\mathbf{z}^{\wedge k} = \mathbf{z}^{\wedge k_{\{1, \dots, k\}}}$$

Introduce the following notation: $\widehat{K} = \{\bar{\mathbf{k}} = (k_1, \dots, k_n) \in \mathbb{Z}^n \mid k_1 \geq k_2 \geq \dots \geq k_n\}$, $\mathbf{e}_j = (0, \dots, \overset{j}{1}, \dots, 0) \in \mathbb{Z}^n$.

³Note that, obviously, $\mathbf{z}^{\wedge n} = \det_q \mathbf{z}$.

Proposition 4 *The representation space V for $\pi_{\alpha,\beta}$ splits into a sum of simple pairwise non-isomorphic $U_q\mathfrak{k}$ -modules as follows:⁴*

$$V = \bigoplus_{\bar{\mathbf{k}} \in \hat{K}} V_{\bar{\mathbf{k}}}, \quad \text{with } V_{\bar{\mathbf{k}}} = \pi_{\alpha,\beta}(U_q\mathfrak{k}) \cdot v_{\bar{\mathbf{k}}}^h \quad \text{and} \quad v_{\bar{\mathbf{k}}}^h = (z^{\wedge 1})^{k_1-k_2} \dots (z^{\wedge n-1})^{k_{n-1}-k_n} (z^{\wedge n})^{k_n}.$$

Proof. Consider the filtration $V = \bigcup_{k=0}^{\infty} V^{(k)}$ with $V^{(k)} = \mathbb{C}[\text{Mat}_n]_q \cdot (\det_q \mathbf{z})^{-k}$. It is sufficient to prove that

$$V^{(k)} = \bigoplus_{k_n \geq -k} V_{\bar{\mathbf{k}}}.$$

Equip the vector space $V^{(k)}$ with the natural grading $V^{(k)} = \bigoplus_{j=-nk}^{\infty} (V^{(k)})_j$ as follows:

$$(V^{(k)})_j = \{v \in V^{(k)} \mid K_0 v = q^{2j} v\}, \quad \text{with } K_0 \stackrel{\text{def}}{=} K_1 K_2^2 K_3^3 \dots K_n^n K_{n+1}^{n-1} \dots K_{2n-2}^2 K_{2n-1}.$$

Therefore we must prove that

$$(\mathbb{C}[\text{Mat}_n]_q \cdot (\det_q \mathbf{z})^{-k})_j = \bigoplus_{\substack{k_n \geq -k, \\ k_1 + \dots + k_n = j}} V_{\bar{\mathbf{k}}}. \quad (7)$$

For $k = 0$ statement (7) means that $(\mathbb{C}[\text{Mat}_n]_q)_j = \bigoplus_{k_n \geq 0, k_1 + \dots + k_n = j} V_{\bar{\mathbf{k}}}.$

First, the dimensions of homogeneous components $\mathbb{C}[\text{Mat}_n]_{q,j}$ of the standardly graded algebra $\mathbb{C}[\text{Mat}_n]_q$ are equal to the dimensions in the classical case:

$$\dim \mathbb{C}[\text{Mat}_n]_{q,j} = \binom{n^2 + j - 1}{j}$$

(it can be easily proved via the Bergman diamond lemma [15], sec. 4.1.5). Secondly, the dimensions of the $U_q\mathfrak{k}$ -modules $V_{\bar{\mathbf{k}}}$ are equal to the classical ones (this follows from results of quantum groups theory [17], chap. 5). Thirdly, there is the well-known Hua result on the coincidence of the dimensions $\mathbb{C}[\text{Mat}_n]_j$ and $\bigoplus_{\substack{k_n \geq 0, \\ k_1 + k_2 + \dots + k_n = j}} V_{\bar{\mathbf{k}}}$ in the classical case [18].

Hence,

$$\dim(\mathbb{C}[\text{Mat}_n]_q)_j = \sum_{\substack{k_n \geq 0, \\ k_1 + \dots + k_n = j}} \dim V_{\bar{\mathbf{k}}},$$

and, finally,

$$(\mathbb{C}[\text{Mat}_n]_q)_j = \bigoplus_{\substack{k_n \geq 0, \\ k_1 + \dots + k_n = j}} V_{\bar{\mathbf{k}}}.$$

For $k > 0$ one has

$$\begin{aligned} (\mathbb{C}[\text{Mat}_n]_q \cdot (\det_q \mathbf{z})^{-k})_j &= \mathbb{C}[\text{Mat}_n]_{q,nk+j} \cdot (\det_q \mathbf{z})^{-k} \\ &= \bigoplus_{\substack{k_n \geq 0, \\ k_1 + \dots + k_n = nk+j}} V_{\bar{\mathbf{k}}} \cdot (\det_q \mathbf{z})^{-k} = \bigoplus_{\substack{k_n \geq -k, \\ k_1 + \dots + k_n = j}} V_{\bar{\mathbf{k}}}. \end{aligned}$$

⁴These isotypic components are $U_q\mathfrak{k}_{ss}$ -isomorphic. However, they are not $U_q\mathfrak{k}$ -isomorphic, since the action of $\pi_{\alpha,\beta}(K_n)$ depends on α, β .

(Since there is no zero divisors in $\mathbb{C}[\text{Mat}_n]_{q, \det_q \mathbf{z}}$, the proof of statement (7) follows from the last equality.) \square

Remark. It can be easily verified that $v_{\mathbf{k}}^h$ is a $U_q \mathfrak{k}$ -highest vector and with weight $(k_1 - k_2, \dots, k_{n-1} - k_n, 2k_n + \alpha - \beta, k_{n-1} - k_n, \dots, k_1 - k_2)$. Then the highest weight of simple $U_q \mathfrak{k}$ -module $V_{\mathbf{k}}$ is equal to $(k_1 - k_2, \dots, k_{n-1} - k_n, 2k_n + \alpha - \beta, k_{n-1} - k_n, \dots, k_1 - k_2)$.

In the classical case $\mathfrak{sl}_{2n} = \mathfrak{p}^- \oplus \mathfrak{k} \oplus \mathfrak{p}^+$, where

$$\mathfrak{p}^- = \left\{ \begin{pmatrix} 0 & 0 \\ A & 0 \end{pmatrix} \middle| A \in \text{Mat}_{n,n}(\mathbb{C}) \right\}, \quad \mathfrak{p}^+ = \left\{ \begin{pmatrix} 0 & A \\ 0 & 0 \end{pmatrix} \middle| A \in \text{Mat}_{n,n}(\mathbb{C}) \right\}.$$

Therefore $U\mathfrak{sl}_{2n} \simeq U\mathfrak{p}^- \otimes U\mathfrak{k} \otimes U\mathfrak{p}^+$ as $U\mathfrak{k}$ -modules ($U\mathfrak{p}^-$ and $U\mathfrak{p}^+$ are $U\mathfrak{k}$ -modules under the adjoint action).

In the quantum case we have an analogue of this decomposition obtained by Jakobsen in [19]. A quantum analogue $\text{ad}_a, a \in U_q \mathfrak{sl}_{2n}$ of the adjoint action is introduced via the Hopf algebra structure of $U_q \mathfrak{sl}_{2n}$. There are n^2 -dimensional vector subspaces $\mathfrak{p}_q^+ = U_q \mathfrak{k} \cdot E_n$, $\mathfrak{p}_q^- = U_q \mathfrak{k} \cdot (K_n F_n)$, which are $U_q \mathfrak{k}$ -invariant under the adjoint action.⁵ Instead of $U\mathfrak{p}^-, U\mathfrak{p}^+$, there are the subalgebras $U_q \mathfrak{p}^-, U_q \mathfrak{p}^+ \subset U_q \mathfrak{sl}_{2n}$ generated by $\mathfrak{p}_q^-, \mathfrak{p}_q^+$, respectively. The algebras $U_q \mathfrak{p}^-$ and $U_q \mathfrak{p}^+$ are $U_q \mathfrak{k}$ -modules under the adjoint action. Therefore in the quantum case we get $U_q \mathfrak{sl}_{2n} \simeq U_q \mathfrak{p}^- \otimes U_q \mathfrak{k} \otimes U_q \mathfrak{p}^+$ as $U_q \mathfrak{k}$ -modules (see [19]). It's worthwhile to note that $U_q \mathfrak{p}^-$ and $U_q \mathfrak{p}^+$ are not Hopf subalgebras unlike the classical case.

In the last part of this section we describe how each $U_q \mathfrak{k}$ -isotypic component $V_{\mathbf{k}}$ transforms under the action of \mathfrak{p}_q^- and \mathfrak{p}_q^+ . This allows one to understand how V transforms under the $U_q \mathfrak{sl}_{2n}$ -action. Since $U_q \mathfrak{k} = U_q \mathfrak{s}(\mathfrak{gl}_n \times \mathfrak{gl}_n) \simeq \mathbb{C}[K_0^{\pm 1}] \otimes (U_q \mathfrak{sl}_n \otimes U_q \mathfrak{sl}_n) = \mathbb{C}[K_0^{\pm 1}] \otimes U_q \mathfrak{k}_{ss}$, where

$$K_0 = K_1 K_2^2 K_3^3 \dots K_n^n K_{n+1}^{n-1} \dots K_{2n-2}^2 K_{2n-1}, \quad (8)$$

and $\pi_{\alpha, \beta}(K_0)$ acts by scalar multiplications in every isotypic component, we see that $V_{\mathbf{k}}$ is a simple $U_q \mathfrak{sl}_n \otimes U_q \mathfrak{sl}_n$ -module. Hence the $U_q \mathfrak{sl}_n \otimes U_q \mathfrak{sl}_n$ -module $V_{\mathbf{k}}$ decomposes into a tensor product of $U_q \mathfrak{sl}_n$ -modules: $V_{\mathbf{k}} \simeq V_{\mathbf{k}}^{(1)} \otimes V_{\mathbf{k}}^{(2)}$ with $V_{\mathbf{k}}^{(1)} = L(k_1 - k_2, \dots, k_{n-1} - k_n)$, $V_{\mathbf{k}}^{(2)} = L^*(k_1 - k_2, \dots, k_{n-1} - k_n)$, where we denote by $L(k_1 - k_2, \dots, k_{n-1} - k_n)$ and $L^*(k_1 - k_2, \dots, k_{n-1} - k_n)$ the simple finite dimensional $U_q \mathfrak{sl}_n$ -modules with highest weights $(k_1 - k_2, \dots, k_{n-1} - k_n)$ and $(k_{n-1} - k_n, \dots, k_1 - k_2)$, respectively.

We can equip the vector spaces $V_{\mathbf{k}}^{(1)}$ and $V_{\mathbf{k}}^{(2)}$ with the structure of $U_q \mathfrak{sl}_n \otimes U_q \mathfrak{sl}_n$ -modules as follows:

$$(\xi \otimes \eta)(v) = \xi \cdot (\varepsilon(\eta)v), \quad (\xi \otimes \eta)(v^*) = \eta \cdot (\varepsilon(\xi)v^*),$$

for all $\xi, \eta \in U_q \mathfrak{sl}_n$, $v \in V_{\mathbf{k}}^{(1)}$, $v^* \in V_{\mathbf{k}}^{(2)}$, where ε denotes the counit of $U_q \mathfrak{sl}_n$. Note that as $U_q \mathfrak{sl}_n \otimes U_q \mathfrak{sl}_n$ -modules

$$\mathfrak{p}_q^+ \simeq \mathbb{C}^n \otimes (\mathbb{C}^n)^*, \quad \mathfrak{p}_q^- \simeq (\mathbb{C}^n)^* \otimes \mathbb{C}^n,$$

where \mathbb{C}^n is the vector representation of $U_q \mathfrak{sl}_n$. Consider the natural maps

$$m_{\mathbf{k}}^+ : \mathfrak{p}_q^+ \otimes V_{\mathbf{k}} \longrightarrow V, \quad m_{\mathbf{k}}^- : \mathfrak{p}_q^- \otimes V_{\mathbf{k}} \longrightarrow V.$$

⁵The operator ad_a is defined on $b \in U_q \mathfrak{sl}_{2n}$ in the following way: $\text{ad}_a(b) = \sum S(a')ba''$, where $\Delta a = \sum a' \otimes a''$ is the comultiplication, S is the antipode in $U_q \mathfrak{sl}_{2n}$.

Since there exist the $U_q \mathfrak{sl}_n \otimes U_q \mathfrak{sl}_n$ -homomorphisms

$$\begin{aligned} \mathfrak{p}_q^+ \otimes V_{\bar{\mathbf{k}}} &\simeq \mathbb{C}^n \otimes (\mathbb{C}^n)^* \otimes V_{\bar{\mathbf{k}}}^{(1)} \otimes V_{\bar{\mathbf{k}}}^{(2)} \simeq \mathbb{C}^n \otimes V_{\bar{\mathbf{k}}}^{(1)} \otimes (\mathbb{C}^n)^* \otimes V_{\bar{\mathbf{k}}}^{(2)}, \\ \mathfrak{p}_q^- \otimes V_{\bar{\mathbf{k}}} &\simeq (\mathbb{C}^n)^* \otimes \mathbb{C}^n \otimes V_{\bar{\mathbf{k}}}^{(1)} \otimes V_{\bar{\mathbf{k}}}^{(2)} \simeq (\mathbb{C}^n)^* \otimes V_{\bar{\mathbf{k}}}^{(1)} \otimes \mathbb{C}^n \otimes V_{\bar{\mathbf{k}}}^{(2)}, \end{aligned}$$

we have the well-defined morphisms

$$\begin{aligned} \mathcal{M}_{\bar{\mathbf{k}}}^+ &: \mathbb{C}^n \otimes V_{\bar{\mathbf{k}}}^{(1)} \otimes (\mathbb{C}^n)^* \otimes V_{\bar{\mathbf{k}}}^{(2)} \longrightarrow V, \\ \mathcal{M}_{\bar{\mathbf{k}}}^- &: (\mathbb{C}^n)^* \otimes V_{\bar{\mathbf{k}}}^{(1)} \otimes \mathbb{C}^n \otimes V_{\bar{\mathbf{k}}}^{(2)} \longrightarrow V. \end{aligned}$$

For instance, if we consider a $U_q \mathfrak{sl}_n$ -highest vector $v_1 \in \mathbb{C}^n \otimes V_{\bar{\mathbf{k}}}^{(1)}$ and a $U_q \mathfrak{sl}_n$ -highest vector $v_2 \in (\mathbb{C}^n)^* \otimes V_{\bar{\mathbf{k}}}^{(2)}$, then $\mathcal{M}_{\bar{\mathbf{k}}}^+(v_1 \otimes v_2)$ is a $U_q \mathfrak{sl}_n \otimes U_q \mathfrak{sl}_n$ -highest vector (or, equivalently, a $U_q \mathfrak{k}$ -highest vector) in V .

In the sequel we are going to get explicit formulas for $U_q \mathfrak{sl}_n$ -highest vectors $\zeta_j \in \mathbb{C}^n \otimes L(k_1 - k_2, \dots, k_{n-1} - k_n)$, $j = 1, \dots, n$ with weights $(k_1, \dots, k_j + 1, \dots, k_n)$, respectively.

In the classical case auxiliary elements F_{mj} of the universal enveloping algebra $U \mathfrak{sl}_n$ are used in such formulas.

Lemma 1 ([8], lemma 3.4) *Let $1 \leq k \leq n - 1$ and $1 \leq m < j \leq n$.*

1. *If $1 \leq k < m$ or $j < k \leq n$, then $E_k F_{mj} = F_{mj} E_k$.*
2. *If $k = m$, then $E_m F_{mj} \equiv F_{m+1,j} (H_m + \dots + H_{j-1} + j - m - 1) \pmod{U \mathfrak{sl}_n \cdot E_m}$.*
3. *If $m < k \leq j$, then $E_k F_{mj} \equiv 0 \pmod{U \mathfrak{sl}_n \cdot E_k}$.* □

Explicit formulas for the elements F_{mj} are used for the proof of lemma.

$$F_{mj} = F_{m+1,j} F_m + \sum_{t=m+2}^j (-1)^{t+m+1} F_{tj} \operatorname{ad}_{F_{t-1}} \dots \operatorname{ad}_{F_m} H(j; m+1, t-1),$$

where $H(j; p, s) = \prod_{a=p}^s (H_a + \dots + H_{j-1} + j - a)$.

We find quantum analogues of the previous lemma and the elements in $U_q \mathfrak{sl}_n$.

For $1 \leq m \leq j \leq n$ define $F_{mj} \in U_q \mathfrak{sl}_n$ inductively as follows:

$$F_{jj} = 1, \quad F_{j-1,j} = F_{j-1} K_{j-1},$$

$$F_{mj} = F_{m+1,j} F_m K_m + \sum_{s=m+2}^j (-1)^{s+m+1} F_{sj} \operatorname{ad}_{F_{s-1}} \dots \operatorname{ad}_{F_{m+1}} (F_m K_m) K(j, m+1, s-1),$$

where $K(j, p, r) = \prod_{a=p}^r q^{j-a} K_a \dots K_{j-1} [H_a + \dots + H_{j-1} + j - a]_q$.

Here and everywhere below we use the standard notation $[x]_q = \frac{q^x - q^{-x}}{q - q^{-1}}$.

Lemma 2 *The following relations are satisfied:*

$$1. K_i F_{mj} = F_{mj} K_i \quad \text{for } 1 \leq i < m-1 \text{ or } j < i \leq n. \quad (9)$$

$$2. K_j F_{mj} = q F_{mj} K_j, \quad K_{m-1} F_{mj} = q F_{mj} K_{m-1}. \quad (10)$$

$$3. q K_{j-1} F_{mj} = F_{mj} K_{j-1}, \quad q K_m F_{mj} = F_{mj} K_m. \quad (11)$$

$$4. E_i F_{mj} = F_{mj} E_i \quad \text{for } 1 \leq i < m-1 \text{ or } j < i \leq n. \quad (12)$$

$$5. E_{m-1} F_{mj} = q F_{mj} E_{m-1}, \quad E_j F_{mj} = q F_{mj} E_j. \quad (13)$$

$$6. E_i F_{mj} \equiv 0 \pmod{U_q \mathfrak{sl}_n \cdot E_i} \quad \text{for } m < i < j. \quad (14)$$

$$7. E_m F_{mj} \equiv F_{m+1,j} q^{j-m} K_m \dots K_{j-1} [H_m + \dots + H_{j-1} + j - m - 1]_q \pmod{U_q \mathfrak{sl}_n \cdot E_m}. \quad (15)$$

This lemma is proved in Appendix. \square

Let $(\varepsilon_1, \dots, \varepsilon_n)$ be the standard basis for \mathbb{C}^n . Suppose $u \in L(k_1 - k_2, \dots, k_{n-1} - k_n)$ is a $U_q \mathfrak{sl}_n$ -highest vector with weight $(k_1 - k_2, \dots, k_{n-1} - k_n)$.

Proposition 5 *Define vectors $\{\zeta_j\}_{j=1}^n$ as follows:*

$$\zeta_j = \sum_{m=1}^j (-q^2)^{m-1} \varepsilon_m \otimes F_{mj} K_-(j, 1, m-1) u \in \mathbb{C}^n \otimes L(k_1 - k_2, \dots, k_{n-1} - k_n),$$

where $K_-(j, p, r) = \prod_{a=p}^r q^{j-a-1} K_a \dots K_{j-1} [H_a + \dots + H_{j-1} + j - a - 1]_q$. Then ζ_j is a $U_q \mathfrak{sl}_n$ -highest vector⁶ with weight $(k_1 - k_2, \dots, k_{j-1} - k_j - 1, k_j + 1 - k_{j+1}, \dots, k_{n-1} - k_n)$ for $j = 1, \dots, n$.

Proof. Using Lemma 2, it is easy to prove that ζ_j are weight vectors. We claim that $E_i \zeta_j = 0$ for all $1 \leq i \leq n$, $1 \leq j \leq n$. Indeed, by Lemma 2

$$\begin{aligned} E_i \zeta_j &= E_i \left(\sum_{m=1}^j (-q^2)^{m-1} \varepsilon_m \otimes F_{mj} K_-(j, 1, m-1) u \right) \\ &= \sum_{m=1}^j (-q^2)^{m-1} E_i (\varepsilon_m \otimes F_{mj} K_-(j, 1, m-1) u) = (-q^2)^i \varepsilon_i \otimes F_{i+1,j} K_-(j, 1, i) u \\ &\quad + (-q^2)^{i-1} E_i (\varepsilon_i \otimes F_{i,j} K_-(j, 1, i-1) u) = (-q^2)^{i-1} (-q^2 \varepsilon_i \otimes F_{i+1,j} K_-(j, 1, i) u \\ &\quad + q \varepsilon_i \otimes F_{i+1,j} q^{j-i} K_i \dots K_{j-1} [H_i + \dots + H_{j-1} + j - i - 1]_q K_-(j, 1, i-1) u) = 0. \quad \square \end{aligned}$$

Similarly, we are going to get explicit formulas for $U_q \mathfrak{sl}_n$ -highest vectors $\xi_j \in (\mathbb{C}^n)^* \otimes L^*(k_1 - k_2, \dots, k_{n-1} - k_n)$. For $1 \leq r \leq t \leq n$ introduce the elements $S_{rt} \in U_q \mathfrak{sl}_n$ as follows:⁷

$$\begin{aligned} S_{tt} &= 1, \quad S_{t-1,t} = F_t K_t, \\ S_{rt} &= S_{r,t-1} F_t K_t + \sum_{s=r+1}^{t-1} S_{r,s-1} \text{ad}_{F_s} \dots \text{ad}_{F_{t-1}} (F_t K_t) L(t, s, t-1), \end{aligned}$$

where $L(j, p, r) = \prod_{a=p}^r q^{a-j} K_{j+1} \dots K_a [H_{j+1} + \dots + H_a + a - j]_q$.

⁶I.e. $E_i \zeta_j = 0$ for all $i = 1, \dots, n-1$.

⁷Classic analogues of these elements were investigated by Lee in [8].

Lemma 3 *The following relations are satisfied:*

1. $K_i S_{rt} = S_{rt} K_i$ for $1 \leq i < r$ or $t+1 < i \leq n$.
2. $K_r S_{rt} = q K_r S_{rt}$, $K_{t+1} S_{rt} = q K_{t+1} S_{rt}$.
3. $K_{r+1} S_{rt} = q^{-1} K_{r+1} S_{rt}$, $K_t S_{rt} = q^{-1} K_t S_{rt}$.
4. $E_i S_{rt} = S_{rt} E_i$ for $1 \leq i < r$ or $t+1 < i \leq n$.
5. $E_r S_{rt} = q S_{rt} E_r$, $E_{t+1} S_{rt} = q S_{rt} E_{t+1}$.
6. $E_i S_{rt} \equiv 0 \pmod{U_q \mathfrak{sl}_n \cdot E_i}$ for $r < i < t$.
7. $E_t S_{rt} \equiv -S_{r,t-1} q^{t-r} K_{r+1} \dots K_t [H_{r+1} + \dots + H_t + t - r - 1]_q \pmod{U_q \mathfrak{sl}_n \cdot E_t}$.

The proof of this lemma is completely analogous to the proof of Lemma 2.

Let $(\varepsilon_1^*, \dots, \varepsilon_n^*)$ be the basis for $(\mathbb{C}^n)^*$ dual to the basis $(\varepsilon_1, \dots, \varepsilon_n)$ for \mathbb{C}^n . Suppose that $u^* \in L^*(k_1 - k_2, \dots, k_{n-1} - k_n)$ is a $U_q \mathfrak{sl}_n$ -highest vector with weight $(k_{n-1} - k_n, \dots, k_1 - k_2)$. The proof of the next statement is similar to the proof of Proposition 5.

Proposition 6 *Define vectors $\{\xi_j\}_{j=1}^n$ as follows:*

$$\xi_j = \sum_{m=j}^n \varepsilon_m^* \otimes S_{jm} L_-(j, m+1, n) u^* \in (\mathbb{C}^n)^* \otimes L^*(k_1 - k_2, \dots, k_{n-1} - k_n)$$

where $L_-(j, p, r) = \prod_{a=p}^r q^{a-j-1} K_{j+1} \dots K_a [H_{j+1} + \dots + H_a + a - j - 1]_q$. Then ξ_j is a $U_q \mathfrak{sl}_n$ -highest vector with weight $(k_{n-1} - k_n, \dots, k_{n-j+1} + 1 - k_{n-j+2}, k_{n-j} - k_{n-j+1} - 1, \dots, k_1 - k_2)$ for $j = 1, \dots, n$. \square

It follows from Propositions 5 and 6 that $\mathcal{M}_{\bar{\mathbf{k}}}^+ : \mathbb{C}^n \otimes V_{\bar{\mathbf{k}}}^{(1)} \otimes (\mathbb{C}^n)^* \otimes V_{\bar{\mathbf{k}}}^{(2)} \rightarrow \bigoplus_{j=1}^n V_{\bar{\mathbf{k}}+\mathbf{e}_j}$. For all $j, k = 1, \dots, n$ the vectors $\mathcal{M}_{\bar{\mathbf{k}}}^+(\zeta_j \otimes \xi_k)$ are $U_q \mathfrak{sl}_n \otimes U_q \mathfrak{sl}_n$ -highest vectors in V . By the action of $\pi_{\alpha, \beta}(K_0)$ (see (8)), the vector $\mathcal{M}_{\bar{\mathbf{k}}}^+(\zeta_j \otimes \xi_k)$ is a $U_q \mathfrak{k}$ -highest vector in $V_{\bar{\mathbf{k}}+\mathbf{e}_j}$ if and only if $k = n - j + 1$. Since every isotypic components occurs with multiplicity one, $\mathcal{M}_{\bar{\mathbf{k}}}^+(\zeta_j \otimes \xi_{n-j+1}) = c_j \cdot v_{\bar{\mathbf{k}}+\mathbf{e}_1}^h = c_j \cdot (z^{\wedge 1})^{k_1 - k_2 + 1} \dots (z^{\wedge n-1})^{k_{n-1} - k_n} (z^{\wedge n})^{k_n}$ for some $c_j \in \mathbb{C}$. (Here and below we suppose that if $\bar{\mathbf{m}} = (m_1, \dots, m_n) \notin \widehat{K}$, then $V_{\bar{\mathbf{m}}} = 0$ and $v_{\bar{\mathbf{m}}}^h = 0$.)

The proof of the next statement, reduced to computation of c_j , is given in Appendix.

Proposition 7 *For every $j = 1, \dots, n$, $\bar{\mathbf{k}} \in \widehat{K}$*

$$\mathcal{M}_{\bar{\mathbf{k}}}^+(\zeta_j \otimes \xi_{n-j+1}) = c_j(\beta, k_j) v_{\bar{\mathbf{k}}+\mathbf{e}_j}^h,$$

where $c_j(\beta, k_j) = q^{-\beta - n/2} [\beta - k_j + j - 1]_q \omega_j(\bar{\mathbf{k}}, q)$ and $\omega_j(\bar{\mathbf{k}}, q) \neq 0$ for all $\bar{\mathbf{k}} \in \widehat{K}$.

We deduce sufficient conditions for reducibility of $\pi_{\alpha, \beta}$ from Proposition 7.

Let α, β be fixed. For any $j = 1, \dots, n$ and $\bar{\mathbf{k}} \in \widehat{K}$ if $c_j(\beta, k_j) \neq 0$, then there exist $v \in V_{\bar{\mathbf{k}}}$, $\xi \in \mathfrak{p}_q^+$ such that $\pi_{\alpha, \beta}(\xi) \cdot v \in V_{\bar{\mathbf{k}}+\mathbf{e}_j}$. That means $\pi_{\alpha, \beta}(U_q \mathfrak{sl}_{2n}) \cdot V_{\bar{\mathbf{k}}} \supset V_{\bar{\mathbf{k}}+\mathbf{e}_j}$.

Let us consider in details other cases, i.e., let $c_j(\beta, k_j) = 0$ for some k_j . For fixed β , by Proposition 7 and (6), the equation $c_j(\beta, k_j) = 0$ is equivalent to $\beta - k_j + j - 1 = 0$.

Corollary 1 For all $j = 1, \dots, n$, $\bar{k} \in \widehat{K}$, the subspace $V_{\leq k}^j \stackrel{\text{def}}{=} \bigoplus_{\{\bar{k}' \in \widehat{K} | k \geq k'\}} V_{\bar{k}'}$ is a $U_q \mathfrak{sl}_{2n}$ -submodule in V iff $\beta - k + j - 1 = 0$.

Proof. Let $j = 1$, the other cases are similar. The necessity easily follows from the above. Prove the sufficiency. If $\beta - k_1 = 0$, then $\mathcal{M}_{\bar{k}}^+(\mathfrak{p}_q^+ \otimes V_{\bar{k}}) \subset \bigoplus_{j=2}^n V_{\bar{k}+\mathbf{e}_j}$. Introduce the natural filtration on $U_q \mathfrak{p}^+$ (here $U_q \mathfrak{p}^+$ is the algebra generated by \mathfrak{p}_q^+) in the following way: $U_q \mathfrak{p}^+ = \bigcup_{n \geq 0} (U_q \mathfrak{p}^+)^{(n)}$. Then $\pi_{\alpha, \beta}((U_q \mathfrak{p}^+)^{(1)})(V_{\bar{k}}) \subset V_{\bar{k}} \oplus (\bigoplus_{j=2}^n V_{\bar{k}+\mathbf{e}_j})$. In the same way, $\pi_{\alpha, \beta}((U_q \mathfrak{p}^+)^{(2)})(V_{\bar{k}}) \subset \pi_{\alpha, \beta}((U_q \mathfrak{p}^+)^{(1)})(V_{\bar{k}} \oplus (\bigoplus_{j=2}^n V_{\bar{k}+\mathbf{e}_j})) \subset V_{\bar{k}} \oplus (\bigoplus_{j=2}^n V_{\bar{k}+\mathbf{e}_j}) \oplus (\bigoplus_{n \geq j_1 \geq j_2 \geq 2} V_{\bar{k}+\mathbf{e}_{j_1}+\mathbf{e}_{j_2}})$. Then $\pi_{\alpha, \beta}(U_q \mathfrak{p}^+)(V_{\bar{k}}) \subset \bigoplus_{m=0}^{\infty} (\bigoplus_{n \geq j_1 \geq \dots \geq j_m \geq 2} V_{\bar{k}+\mathbf{e}_{j_1}+\dots+\mathbf{e}_{j_m}})$, and

$$\begin{aligned} \pi_{\alpha, \beta}(U_q \mathfrak{sl}_{2n})(V_{\bar{k}}) &\subset \pi_{\alpha, \beta}(U_q \mathfrak{p}^-) \pi_{\alpha, \beta}(U_q \mathfrak{k}) (\pi_{\alpha, \beta}(U_q \mathfrak{p}^+) V_{\bar{k}}) \\ &\subset \pi_{\alpha, \beta}(U_q \mathfrak{p}^-) \pi_{\alpha, \beta}(U_q \mathfrak{k}) \left(\bigoplus_{m=0}^{\infty} \left(\bigoplus_{n \geq j_1 \geq \dots \geq j_m \geq 2} V_{\bar{k}+\mathbf{e}_{j_1}+\dots+\mathbf{e}_{j_m}} \right) \right) \\ &\subset \pi_{\alpha, \beta}(U_q \mathfrak{p}^-) \left(\bigoplus_{m \geq 0} \left(\bigoplus_{n \geq j_1 \geq \dots \geq j_m \geq 2} V_{\bar{k}+\mathbf{e}_{j_1}+\dots+\mathbf{e}_{j_m}} \right) \right) \subset V_{\leq k}^1. \end{aligned}$$

Obviously the subspace $V_{\leq k}^1$ is a $U_q \mathfrak{sl}_{2n}$ -submodule in V . \square

By the same arguments as in Propositions 5, 6 and 7, one has

Proposition 8 Define vectors $\{\xi'_j\}_{j=1}^n$ as follows:

$$\xi'_j = \sum_{m=j}^n \varepsilon_m^* \otimes S_{jm} L_-(j, m+1, n) u \in (\mathbb{C}^n)^* \otimes L(k_1 - k_2, \dots, k_{n-1} - k_n)$$

where $L_-(j, p, r) = \prod_{a=p}^r q^{a-j-1} K_{j+1} \dots K_a [H_{j+1} + \dots + H_a + a - j - 1]_q$. Then ξ'_j is a $U_q \mathfrak{sl}_n$ -highest vector with weight $(k_1 - k_2, \dots, k_{j-1} - k_j + 1, k_j - 1 - k_{j+1}, \dots, k_{n-1} - k_n)$ for $j = 1, \dots, n$. \square

Proposition 9 Define vectors $\{\zeta'_j\}_{j=1}^n$ as follows

$$\zeta'_j = \sum_{m=1}^j (-q^2)^{m-1} \varepsilon_m \otimes F_{mj} K_-(j, 1, m-1) u^* \in \mathbb{C}^n \otimes L^*(k_1 - k_2, \dots, k_{n-1} - k_n)$$

where $K_-(j, p, r) = \prod_{a=p}^r q^{j-a-1} K_a \dots K_{j-1} [H_a + \dots + H_{j-1} + j - a - 1]_q$. Then ζ'_j is a $U_q \mathfrak{sl}_n$ -highest vector with weight $(k_{n-1} - k_n, \dots, k_{n-j+1} - k_{n-j} - 1, k_{n-j} + 1 - k_{n-j-1}, \dots, k_1 - k_2)$ for $j = 1, \dots, n$. \square

The proof of the next statement, reduced as for Proposition 7 to computation of d_j , is given in Appendix.

Proposition 10 For every $j = 1, \dots, n$, $\bar{\mathbf{k}} \in \widehat{K}$

$$\mathcal{M}^{-}_{\bar{\mathbf{k}}}(\xi'_j \otimes \zeta'_{n-j+1}) = d_j(\alpha, k_j) v_{\bar{\mathbf{k}} - \mathbf{e}_j}^h,$$

where $d_j(\alpha, k_j) = q^{\alpha+n/2}[\alpha + k_j + n - j]_q \varpi_j(\bar{\mathbf{k}}, q)$ and $\varpi_j(\bar{\mathbf{k}}, q) \neq 0$ for all $\bar{\mathbf{k}} \in \widehat{K}$. \square

By (6) and Proposition 10, we see that the equations $d_j(\alpha, k_j) = 0$ and $\alpha + k_j + n - j = 0$ are equivalent.

Corollary 2 For all $j = 1, \dots, n$, $\bar{\mathbf{k}} \in \widehat{K}$ the subspace $V_{\geq k}^j \stackrel{\text{def}}{=} \bigoplus_{\{\bar{\mathbf{k}}' \in \widehat{K} | k'_j \geq k\}} V_{\bar{\mathbf{k}}'}$ is a $U_q \mathfrak{sl}_{2n}$ -submodule in V iff $\alpha + k_j + n - j = 0$. \square

V Reducibility of $\pi_{\alpha, \beta}$

Proposition 11 The representation $\pi_{\alpha, \beta}$ is irreducible if and only if α, β satisfy the following equivalent conditions:⁸

1. $\alpha \notin \mathbb{Z}$;
2. $\beta \notin \mathbb{Z}$.

Proof. Suppose $\alpha \notin \mathbb{Z}$, $\beta \notin \mathbb{Z}$. Consider the system of equations

$$\begin{cases} \beta - k_1 = 0, \\ \beta - k_2 + 1 = 0, \\ \dots\dots\dots \\ \beta - k_n + n - 1 = 0, \\ \alpha + k_1 + n - 1 = 0, \\ \dots\dots\dots \\ \alpha + k_n = 0. \end{cases}$$

This system has no integral solution. Therefore $c_j(\beta, k_j)$ and $d_j(\alpha, k_j)$ do not vanish. Let W be a $U_q \mathfrak{sl}_{2n}$ -submodule of V . Then $W = \bigoplus_{\bar{\mathbf{k}} \in I} V_{\bar{\mathbf{k}}}$ for some $I \subset \widehat{K}$. Then, for all $\bar{\mathbf{k}} \in I$

and $j = 1, \dots, n$, it follows that $\bar{\mathbf{k}} + \mathbf{e}_j, \bar{\mathbf{k}} - \mathbf{e}_j \in I$ (if the respective indexes belong to \widehat{K}). Therefore if $I \neq \emptyset$, then $I = \widehat{K}$, and the module V have no proper submodules, i.e. it is simple. Conversely, by Corollaries 1 and 2, if $\pi_{\alpha, \beta}$ is irreducible, then $\alpha \notin \mathbb{Z}$, $\beta \notin \mathbb{Z}$. \square

Corollary 3 Let $\alpha, \beta \in \mathbb{Z}$, and let W be the representation space of a subrepresentation of $\pi_{\alpha, \beta}$. Then W is a finite intersection of some of the $U_q \mathfrak{sl}_{2n}$ -modules $V_{\geq k}^j, V_{\leq k}^j$ defined in Corollaries 1, 2.

The proof follows directly from the previous proof. \square

Now suppose that $\alpha, \beta \in \mathbb{Z}$. We will investigate reducibility and proper subrepresentations of $\pi_{\alpha, \beta}$. We use figures as in [1, 8] for description.

⁸Since $\alpha - \beta \in \mathbb{Z}$, these conditions are equivalent.

Each $U_q\mathfrak{k}$ -isotypic component $V_{\bar{\mathbf{k}}}$ is assigned to the point $(k_1, \dots, k_n) \in \mathbb{R}^n$. Thus \widehat{K} is assigned to the set $\mathbf{K}^+ = \{(k_1, \dots, k_n) \mid k_1 \geq \dots \geq k_n\} \subset \mathbb{R}^n$. Consider $2n$ hyperplanes:

$$\mathcal{L}_j^+ : k_j = \beta + j - 1; \quad \mathcal{L}_j^- : k_j = -\alpha - n + j.$$

These hyperplanes are parallel to the coordinate axis and pass through points with integral coordinates. The distance between \mathcal{L}_j^+ and \mathcal{L}_j^- is equal to $\alpha + \beta + n - 1$.

By Corollaries 1 and 2,

$$\bar{\mathbf{k}} \in \mathcal{L}_j^+ \quad \text{iff} \quad U_q\mathfrak{sl}_{2n} \cdot V_{\bar{\mathbf{k}}} \not\subset V_{\bar{\mathbf{k}}+\mathbf{e}_j}; \quad \bar{\mathbf{k}} \in \mathcal{L}_j^- \quad \text{iff} \quad U_q\mathfrak{sl}_{2n} \cdot V_{\bar{\mathbf{k}}} \not\subset V_{\bar{\mathbf{k}}-\mathbf{e}_j}.$$

Investigate the example $n = 2$. In this case \mathcal{L}_j^\pm , $j = 1, 2$ are just lines on the plane \mathbb{R}^2 , parallel to the coordinate axis. Let us consider different values of $\alpha + \beta$.

Case 1. $\alpha + \beta \geq 0$. In this case the line \mathcal{L}_1^+ lies to the right of \mathcal{L}_1^- , \mathcal{L}_2^+ lies higher than \mathcal{L}_2^- . The lines $\mathcal{L}_1^\pm, \mathcal{L}_2^\pm$ are shown on Fig.1. The intersection point of \mathcal{L}_1^+ and \mathcal{L}_2^- has the coordinates $(\beta, -\alpha)$ and belongs to \mathbf{K}^+ . Arrows attached to \mathcal{L}_j^\pm show the direction of isotypic components "movement" under $\pi_{\alpha, \beta}$. There exists a unique simple submodule $V^s = \bigoplus_{\{\bar{\mathbf{k}} \in \widehat{K} \mid k_1 \leq \beta, k_2 \geq -\alpha\}} V_{\bar{\mathbf{k}}}$ in V .

Case 2. $\alpha + \beta = -1$. In this case the lines \mathcal{L}_1^+ and \mathcal{L}_1^- , \mathcal{L}_2^+ and \mathcal{L}_2^- coincide. The intersection point of the lines \mathcal{L}_1^+ and \mathcal{L}_2^+ does not belong to \mathbf{K}^+ (Fig.2). There are two simple submodules in V : $V_1^s = \bigoplus_{\{\bar{\mathbf{k}} \in \widehat{K} \mid k_1 = -1 - \alpha\}} V_{\bar{\mathbf{k}}}$ and $V_2^s = \bigoplus_{\{\bar{\mathbf{k}} \in \widehat{K} \mid k_2 = -\alpha\}} V_{\bar{\mathbf{k}}}$.

Case 3. $\alpha + \beta = -2$. In this case the line \mathcal{L}_1^+ lies to the left of \mathcal{L}_1^- , \mathcal{L}_2^+ lies lower than \mathcal{L}_2^- . However, the lines \mathcal{L}_1^- and \mathcal{L}_2^+ intersect in the point with coordinates $(-\alpha - 1, \beta + 1)$ (see Fig.3). Besides, the distance between \mathcal{L}_j^+ and \mathcal{L}_j^- is equal to 1. This shows that V is a direct sum of three submodules:

$$V_1^s = \bigoplus_{\{\bar{\mathbf{k}} \in \widehat{K} \mid k_1 \leq \beta\}} V_{\bar{\mathbf{k}}}, \quad V_2^s = \bigoplus_{\{\bar{\mathbf{k}} \in \widehat{K} \mid k_2 \geq -\alpha\}} V_{\bar{\mathbf{k}}}, \quad V_3^s = \bigoplus_{\{\bar{\mathbf{k}} \in \widehat{K} \mid k_1 \geq -\alpha - 1, k_2 \leq \beta + 1\}} V_{\bar{\mathbf{k}}}.$$

Case 4. $\alpha + \beta \leq -3$. In this case the intersection point of \mathcal{L}_1^+ and \mathcal{L}_1^- belongs to \mathbf{K}^+ (see Fig.4). Also, there are simple submodules V_1^s, V_2^s, V_3^s in V , but V does not decompose into their direct sum.

Turn now to the general case. Consider all possible values of $\alpha + \beta + n - 1$.

Case 1. $\alpha + \beta + n - 1 \geq 1$. In this case the hyperplanes \mathcal{L}_j^\pm , $j = 1, \dots, n$ bound in \mathbf{K}^+ the subset that corresponds to a unique simple *finite dimensional* submodule

$$V^s = \bigoplus_{\{\bar{\mathbf{k}} \in \widehat{K} \mid -\alpha - n + j \leq k_j \leq \beta + j - 1 \text{ for all } j = 1, \dots, n\}} V_{\bar{\mathbf{k}}}.$$

Case 2. $\alpha + \beta + n - 1 = 0$. In this case the hyperplanes \mathcal{L}_j^+ and \mathcal{L}_j^- coincide. There are n simple submodules in V :

$$V_j^s = \bigoplus_{\{\bar{\mathbf{k}} \in \widehat{K} \mid k_j = \beta + j - 1\}} V_{\bar{\mathbf{k}}}, \quad j = 1, \dots, n. \quad (16)$$

Case 3. $\alpha + \beta = -n$. Here the distance between \mathcal{L}_j^+ and \mathcal{L}_j^- is equal to 1. This allows one to decompose the set \widehat{K} into a direct sum of $n + 1$ subsets \widehat{K}_i , $i = 1, \dots, n + 1$, those

correspond to the simple submodules: $V_i^s = \bigoplus_{\{\bar{\mathbf{k}} \in \widehat{K}_i\}} V_{\bar{\mathbf{k}}} \subset V$. The subsets \widehat{K}_i are defined as follows:

$$\widehat{K}_i = \{\bar{\mathbf{k}} \in \widehat{K} | k_{i-1} \geq -\alpha - n + i - 1, \beta + i - 1 \geq k_i\}$$

(for $i = 1$ and $i = n + 1$ we put respectively $\widehat{K}_1 = \{\bar{\mathbf{k}} \in \widehat{K} | k_1 \leq \beta\}$ and $\widehat{K}_{n+1} = \{\bar{\mathbf{k}} \in \widehat{K} | k_n \geq -\alpha\}$).

Remark. Since $k_j \leq \beta + j - 1$ and $k_j \geq k_l$ for all $j \leq l \leq n$, we see that $k_l \leq \beta + l - 1$. By the same reason, since $k_j \leq -\alpha - n + j$ and $k_j \geq k_l$ for all $j \geq l \geq 1$, we see that $k_l \leq -\alpha - n + l$.

Case 4. $\alpha + \beta + n - 1 \leq -2$. Also, there are simple submodules corresponded to the subsets \widehat{K}_i . However, V is not equal to their direct sum.

Thus we have proved the following

Corollary 4 *For $\alpha, \beta \in \mathbb{Z}$ the only one from the representations $\pi_{\alpha, \beta}$ and $\pi_{-n-\beta, -n-\alpha}$ has an irreducible finite dimensional subrepresentation.* \square

VI Intertwining operators

In this section we construct the intertwining operators between the representations $\pi_{\alpha, \beta}$ and $\pi_{-n-\beta, -n-\alpha}$ for non-integral α, β . This allows one to prove Proposition 2.

Let $A : V \rightarrow V$ be an intertwining operator, i.e., for all $\xi \in U_q \mathfrak{sl}_{2n}$, $v \in V$, we have $A\pi_{\alpha, \beta}(\xi)(v) = \pi_{-n-\beta, -n-\alpha}(\xi)(Av)$. The operators $\pi_{\alpha, \beta}(U_q \mathfrak{k}_{ss})$ are independent of α, β and $\pi_{\alpha, \beta}(K_n) = \pi_{-n-\beta, -n-\alpha}(K_n)$. Also, $V_{\bar{\mathbf{k}}}$ and $V_{\bar{\mathbf{m}}}$ are non-isomorphic $U_q \mathfrak{k}$ -modules for $\bar{\mathbf{k}} \neq \bar{\mathbf{m}}$. Then $A(\alpha, \beta)|_{V_{\bar{\mathbf{k}}}} = a_{\bar{\mathbf{k}}}(\alpha, \beta)$, $a_{\bar{\mathbf{k}}}(\alpha, \beta) \in \mathbb{C}$. Let us find necessary conditions for A to be an intertwining operator in terms of $a_{\bar{\mathbf{k}}}(\alpha, \beta)$. By Propositions 5, 6, 8, and 9, it follows that for all $\bar{\mathbf{k}} \in \widehat{K}$ there exist $\vartheta_j, \eta_j \in U_q \mathfrak{sl}_{2n}$, $j = 1, \dots, n$, such that $\pi_{\alpha, \beta}(\eta_j)(v_{\bar{\mathbf{k}}}^h) = c_j(\beta, k_j)v_{\bar{\mathbf{k}}+\mathbf{e}_j}^h$ and $\pi_{\alpha, \beta}(\vartheta_j)(v_{\bar{\mathbf{k}}}^h) = d_j(\alpha, k_j)v_{\bar{\mathbf{k}}-\mathbf{e}_j}^h$. (Recall that $v_{\bar{\mathbf{k}}}^h$ is the $U_q \mathfrak{k}$ -highest vector in $V_{\bar{\mathbf{k}}}$.) Therefore the necessary conditions look as follows: for all $j = 1, \dots, n, \bar{\mathbf{k}} \in \widehat{K}$,

$$A\pi_{\alpha, \beta}(\eta_j)(v_{\bar{\mathbf{k}}}^h) = \pi_{-n-\beta, -n-\alpha}(\eta_j)(Av_{\bar{\mathbf{k}}}^h) \text{ and } A\pi_{\alpha, \beta}(\vartheta_j)(v_{\bar{\mathbf{k}}}^h) = \pi_{-n-\beta, -n-\alpha}(\vartheta_j)(Av_{\bar{\mathbf{k}}}^h).$$

Equivalently, in terms of $a_{\bar{\mathbf{k}}}$,

$$a_{\bar{\mathbf{k}}+\mathbf{e}_j}(\alpha, \beta)c_j(\beta, k_j)v_{\bar{\mathbf{k}}+\mathbf{e}_j}^h = a_{\bar{\mathbf{k}}}(\alpha, \beta)c_j(-n-\alpha, k_j)v_{\bar{\mathbf{k}}+\mathbf{e}_j}^h,$$

$$a_{\bar{\mathbf{k}}-\mathbf{e}_j}(\alpha, \beta)d_j(\alpha, k_j)v_{\bar{\mathbf{k}}-\mathbf{e}_j}^h = a_{\bar{\mathbf{k}}}(\alpha, \beta)d_j(-n-\beta, k_j)v_{\bar{\mathbf{k}}-\mathbf{e}_j}^h.$$

Thus the coefficients $a_{\bar{\mathbf{k}}}$ of the intertwining operator A must satisfy the following conditions: for all $j = 1, \dots, n, \bar{\mathbf{k}} \in \widehat{K}$,

$$\frac{a_{\bar{\mathbf{k}}+\mathbf{e}_j}(\alpha, \beta)}{a_{\bar{\mathbf{k}}}(\alpha, \beta)} = \frac{c_j(-n-\alpha, k_j)}{c_j(\beta, k_j)}, \quad \frac{a_{\bar{\mathbf{k}}-\mathbf{e}_j}(\alpha, \beta)}{a_{\bar{\mathbf{k}}}(\alpha, \beta)} = \frac{d_j(-n-\beta, k_j)}{d_j(\alpha, k_j)}.$$

We get from Propositions 7 and 10 that for all $j = 1, \dots, n, \bar{\mathbf{k}} \in \widehat{K}$,

$$\frac{a_{\bar{\mathbf{k}}+\mathbf{e}_j}(\alpha, \beta)}{a_{\bar{\mathbf{k}}}(\alpha, \beta)} = q^{n+\alpha+\beta} \frac{[-n-\alpha-k_j+j-1]_q}{[\beta-k_j+j-1]_q}, \quad \frac{a_{\bar{\mathbf{k}}-\mathbf{e}_j}(\alpha, \beta)}{a_{\bar{\mathbf{k}}}(\alpha, \beta)} = q^{-n-\beta-\alpha} \frac{[-\beta+k_j-j]_q}{[\alpha+k_j+n-j]_q}.$$

As we see, the coefficients $a_{\overline{\mathbf{k}}}(\alpha, \beta)$ are defined up to a scalar multiplier. By additional assumption $a_{\overline{\mathbf{0}}}(\alpha, \beta) = 1$, we get the explicit formulas for the coefficients $a_{\overline{\mathbf{k}}}(\alpha, \beta) = A(\alpha, \beta)|V_{\overline{\mathbf{k}}}$ of the intertwining operator A

$$a_{\overline{\mathbf{k}}}(\alpha, \beta) = \prod_{j=1}^n P_j(\alpha, \beta), \quad (17)$$

where

$$P_j(\alpha, \beta) = \begin{cases} \prod_{i=0}^{k_j-1} \frac{1-q^{2(\alpha+n+i-j+1)}}{1-q^{2(-\beta+i-j+1)}}, & \text{for } k_j > 0, \\ 1, & \text{for } k_j = 0, \\ \prod_{i=1-k_j}^0 \frac{1-q^{2(-\beta+i-j)}}{1-q^{2(\alpha+n+i-j)}}, & \text{for } k_j < 0. \end{cases}$$

For fixed $\alpha - \beta \in \mathbb{Z}$, the operator A is a meromorphic operator-function with simple poles in integral points.

VII Unitarizable representations of the degenerate principal series

In this section we find necessary and sufficient conditions for modules of degenerate principal series and their simple submodules to be unitarizable.

Equip $U_q \mathfrak{sl}_{2n}$ with the involution $*$ as follows:

$$\begin{aligned} E_n^* &= -K_n F_n, & F_n^* &= -E_n K_n^{-1}, & K_n^* &= K_n, \\ E_j^* &= K_j F_j, & F_j^* &= E_j K_j^{-1}, & K_j^* &= K_j, & j &= 1, \dots, 2n-1, j \neq n. \end{aligned}$$

The $*$ -Hopf algebra $U_q \mathfrak{su}_{n,n} \stackrel{\text{def}}{=} (U_q \mathfrak{sl}_{2n}, *)$ is a q -analogue of $U \mathfrak{su}_{n,n}$, and its subalgebra $U_q \mathfrak{s}(\mathfrak{u}_n \times \mathfrak{u}_n) \stackrel{\text{def}}{=} (U_q \mathfrak{k}, *)$ is a q -analogue of $U \mathfrak{s}(\mathfrak{u}_n \times \mathfrak{u}_n)$.

Let us introduce two auxiliary $*$ -algebras $\text{Pol}(S(\mathbb{U}))_q$ and $\text{Pol}(\widehat{S(\mathbb{U})})_q$ (a quantum analogue of the Shilov boundary $S(\mathbb{U})$ of the matrix ball is introduced in [20]). Equip the algebra $\mathbb{C}[\text{Mat}_n]_{q, \det_q \mathbf{z}}$ with the involution $*$ defined by the formula

$$(z_a^b)^* = (-q)^{a+b-2n} (\det_q \mathbf{z})^{-1} \det_q \mathbf{z}_a^b,$$

where $\det_q \mathbf{z}_a^b$ is the q -determinant of the matrix derived from \mathbf{z} by deleting the line b and the column a . Put $\text{Pol}(S(\mathbb{U}))_q = (\mathbb{C}[\text{Mat}_n]_{q, \det_q \mathbf{z}}, *)$ and equip it with the natural structure of a $*$ -module algebra over $U_q \mathfrak{su}_{n,n}$. The involutions in $\text{Pol}(S(\mathbb{U}))_q$ and $U_q \mathfrak{su}_{n,n}$ are compatible, i.e., for all $f \in \text{Pol}(S(\mathbb{U}))_q$, $\xi \in U_q \mathfrak{su}_{n,n}$ we have

$$(\xi f)^* = (S(\xi))^* f^*,$$

where S is the antipode in the Hopf algebra $U_q \mathfrak{sl}_{2n}$.

The $*$ -algebra $\text{Pol}(\widehat{S(\mathbb{U})})_q$ is generated by z_a^b , $a, b = 1, \dots, n$, $(\det_q \mathbf{z})^{-1}$, t and t^{-1} . The relations between z_a^b and $(\det_q \mathbf{z})^{-1}$ are inherited from the $*$ -algebra $\text{Pol}(S(\mathbb{U}))_q$, the other relations are provided by the following:

$$t^{-1}t = tt^{-1} = 1, \quad tt^* = t^*t, \quad tz_a^b = q^{-1}z_a^b t, \quad t^*z_a^b = qz_a^b t^*, \quad a, b = 1, \dots, n.$$

Consider an embedding of algebras $\text{Pol}(\widehat{S(\mathbb{U})})_q \hookrightarrow \mathbb{C}[\text{Pl}_{n,2n}]_{q,t}$ which maps t to t and z_a^b to $t^{-1}t_{\{1,\dots,n\}J_{ab}}^{\wedge n}$ (see (3)). Using this embedding, we can extend the $U_q\mathfrak{su}_{n,n}$ -module structure from $\text{Pol}(S(\mathbb{U}))_q$ onto $\text{Pol}(\widehat{S(\mathbb{U})})_q$.

In [20], the invariant integral over the Shilov boundary of the quantum matrix ball $f \mapsto \int_{S(\mathbb{U})_q} f d\mu$ is defined and the following statement is actually proved.

Proposition 12 *The linear subspace $(t^{-n})^* \cdot \text{Pol}(S(\mathbb{U}))_q \cdot t^{-n} \subset \text{Pol}(\widehat{S(\mathbb{U})})_q$ is a $U_q\mathfrak{su}_{n,n}$ -module. The linear functional*

$$(t^{-n})^* \cdot f \cdot t^{-n} \mapsto \int_{S(\mathbb{U})_q} f d\mu$$

is a $U_q\mathfrak{su}_{n,n}$ -invariant integral.

The precise meaning of two next propositions will be given if we continue $\text{Pol}(\widehat{S(\mathbb{U})})_q$ via adding to the list of generators t^λ , $(t^*)^\lambda$, $(\det_q \mathbf{z})^\lambda$ for all $\lambda \in \mathbb{C}$. The relations between the "new" generators and the action of E_j , F_j , $K_j^{\pm 1}$, $j = 1, \dots, 2n-1$ can be derived from the corresponding formulas for t^m , $(\det_q \mathbf{z})^m$ and $(t^*)^m$, where $m \in \mathbb{Z}$. From the previous proposition it follows

Proposition 13 (cf. [13], lemma 3.2) *Let $\text{Re} \lambda = -n$. Then the linear subspace*

$$((\det_q \mathbf{z})^{\lambda/2} t^\lambda)^* \cdot \text{Pol}(S(\mathbb{U}))_q \cdot (\det_q \mathbf{z})^{\lambda/2} t^\lambda \in \text{Pol}(\widehat{S(\mathbb{U})})_q$$

is a $U_q\mathfrak{su}_{n,n}$ -module. The linear functional

$$((\det_q \mathbf{z})^{\lambda/2} t^\lambda)^* \cdot f \cdot (\det_q \mathbf{z})^{\lambda/2} t^\lambda \mapsto \int_{S(\mathbb{U})_q} f d\mu$$

is a $U_q\mathfrak{su}_{n,n}$ -invariant integral.

For each $\alpha, \beta \in \mathbb{Z}$ define an embedding $i_{\alpha,\beta} : V = \mathbb{C}[\text{Mat}_n]_{q,\det_q \mathbf{z}} \hookrightarrow \text{Pol}(\widehat{S(\mathbb{U})})_q$ by the formula $i_{\alpha,\beta}(f) = f \cdot (\det_q \mathbf{z})^\alpha \cdot t^{\alpha+\beta}$ for all $f \in \mathbb{C}[\text{Mat}_n]_{q,\det_q \mathbf{z}}$. Using these embeddings and the commutative relations between t , t^{-1} and $\det_q \mathbf{z}$, we get

Corollary 5 *Let $\text{Re}(\alpha + \beta) = -n$. Then the sesquilinear form $V \times V \rightarrow \mathbb{C}$ defined by*

$$\langle f_1, f_2 \rangle = \int_{S(\mathbb{U})_q} f_2^* f_1 d\mu$$

satisfies the condition $(\pi_{\alpha,\beta}(\xi)u, v) = (u, \pi_{\alpha,\beta}(\xi^)v)$ for all $u, v \in V$, $\xi \in U_q\mathfrak{sl}_{2n}$.*

Recall the definition of unitarizable module. Let A be a $*$ -Hopf algebra, W an A -module. Then an A -module W is *unitarizable* if there exists an Hermitian form⁹ (\cdot, \cdot) , which is A -invariant, i.e.,

$$(au, v) = (u, a^*v) \quad \text{for any } u, v \in W, a \in A.$$

⁹I.e., sesquilinear Hermitian-symmetric positive definite form.

Therefore the representation $\pi_{\alpha,\beta}$ is unitary if $\operatorname{Re}(\alpha + \beta) = -n$. Such representations form *the principal unitary series*.

Now we are going to find all unitarizable simple modules of degenerate principal series and their unitarizable submodules.

Weight subspaces are pairwise orthogonal with respect to every $U_q \mathfrak{su}_{n,n}$ -invariant scalar product. Therefore the isotypic components $V_{\bar{\mathbf{k}}}$ are pairwise orthogonal too. From Proposition 4 and the Burnside theorem (see [21], §27), it follows that in every component $V_{\bar{\mathbf{k}}}$ there exists a unique up to a constant $U_q \mathfrak{s}(\mathfrak{u}_n \times \mathfrak{u}_n)$ -invariant scalar product. Fix such scalar products via the integral over the Shilov boundary of the quantum matrix ball [20]:

$$\langle u, v \rangle_{\bar{\mathbf{k}}} = \int_{S(\mathbb{U})_q} v^* u d\mu \quad u, v \in V_{\bar{\mathbf{k}}}.$$

Hence each invariant scalar product $(\cdot, \cdot) : V \times V \rightarrow \mathbb{C}$ is assigned to a set $\{c_{\bar{\mathbf{k}}}\}_{\bar{\mathbf{k}} \in K^+} \subset \mathbb{R}_+$ such that $(u, v) = c_{\bar{\mathbf{k}}} \langle u, v \rangle_{\bar{\mathbf{k}}}$ for all $u, v \in V_{\bar{\mathbf{k}}}$. Conversely, each $\{c_{\bar{\mathbf{k}}}\}_{\bar{\mathbf{k}} \in K^+} \subset \mathbb{R}_+$ defines a unique sesquilinear Hermitian-symmetric positive definite $U_q \mathfrak{s}(\mathfrak{u}_n \times \mathfrak{u}_n)$ -invariant form in V .

Let us find explicit conditions for the coefficients $\{c_{\bar{\mathbf{k}}}\}$ to define the $U_q \mathfrak{su}_{n,n}$ -invariant form.

Using the decomposition $U_q \mathfrak{sl}_{2n} \simeq U_q \mathfrak{p}^- \otimes U_q \mathfrak{k} \otimes U_q \mathfrak{p}^+$ from Section IV and the definitions of $U_q \mathfrak{p}^+$ and $U_q \mathfrak{p}^-$, we see that it is sufficient to investigate invariance of (\cdot, \cdot) under the subspaces \mathfrak{p}_q^+ and \mathfrak{p}_q^- . Moreover, it is enough to prove \mathfrak{p}_q^+ -invariance of (\cdot, \cdot) . We can see that if (\cdot, \cdot) is \mathfrak{p}_q^+ -invariant, then it is \mathfrak{p}_q^- -invariant. Indeed, for all $\eta \in \mathfrak{p}_q^-$, $u, v \in V$ we have $(\eta u, v) = \overline{(v, \eta u)} = \overline{(\eta^* v, u)} = (u, \eta^* v)$.

Investigate the \mathfrak{p}_q^+ -invariance of the form (\cdot, \cdot) . From results of Section IV it follows that $\pi_{\alpha,\beta}(\mathfrak{p}_q^+)(V_{\bar{\mathbf{k}}}) \subset \bigoplus_{j=1}^n V_{\bar{\mathbf{k}}+\mathbf{e}_j}$. Since the isotypic components $V_{\bar{\mathbf{k}}}$ are pairwise orthogonal, one need to check the invariance in "non-zero cases" only (that means for $u \in V_{\bar{\mathbf{k}}}$, $v \in V_{\bar{\mathbf{k}}+\mathbf{e}_j}$, $j = 1, \dots, n$). In this case the invariant conditions are the following: for all $\xi \in U_q \mathfrak{sl}_{2n}$, $u \in V_{\bar{\mathbf{k}}}$, $v \in V_{\bar{\mathbf{k}}+\mathbf{e}_j}$, $j = 1, \dots, n$,

$$(P_{\bar{\mathbf{k}}+\mathbf{e}_j}(\pi_{\alpha,\beta}(\xi)u), v)|_{V_{\bar{\mathbf{k}}+\mathbf{e}_j}} = (u, P_{\bar{\mathbf{k}}}(\pi_{\alpha,\beta}(\xi^*)v))|_{V_{\bar{\mathbf{k}}}},$$

where $P_{\bar{\mathbf{k}}} : V \longrightarrow V_{\bar{\mathbf{k}}}$ is an orthogonal projection onto $V_{\bar{\mathbf{k}}}$. In other words,

$$c_{\bar{\mathbf{k}}+\mathbf{e}_j} \langle P_{\bar{\mathbf{k}}+\mathbf{e}_j}(\pi_{\alpha,\beta}(\xi)u), v \rangle_{\bar{\mathbf{k}}+\mathbf{e}_j} = c_{\bar{\mathbf{k}}} \langle u, P_{\bar{\mathbf{k}}}(\pi_{\alpha,\beta}(\xi^*)v) \rangle_{\bar{\mathbf{k}}}.$$

First consider the case $\alpha, \beta \notin \mathbb{Z}$. Recall that from Propositions 5, 6, 8, and 9 it follows that in $(\mathfrak{p}_q^- \oplus \mathfrak{p}_q^+) \otimes V_{\bar{\mathbf{k}}}$ there exist $U_q \mathfrak{k}_{ss}$ -highest vectors $\psi_{j,l}^\pm$, $j, l = 1, \dots, n$ with weights $(k_1 - k_2, \dots, k_{j-1} - (k_j \pm e_j), (k_j \pm e_j) - k_{j+1}, \dots, k_{n-1} - k_n, 2k_n + \alpha - \beta, k_{n-1} - k_n, \dots, (k_{n-l+1} \mp e_l) - k_{n-l+2}, k_{n-l} - (k_{n-l+1} \mp e_l), \dots, k_1 - k_2)$, respectively. Define $U_q \mathfrak{k}$ -invariant maps

$$T_{\bar{\mathbf{k}},j}^\pm : (\mathfrak{p}_q^- \oplus \mathfrak{p}_q^+) \otimes V_{\bar{\mathbf{k}}} \longrightarrow V_{\bar{\mathbf{k}} \pm \mathbf{e}_j}$$

by their values on the $U_q \mathfrak{k}_{ss}$ -highest vectors as follows:

$$\begin{aligned} T_{\bar{\mathbf{k}},j}^+(\psi_{j,l}^+) &= \begin{cases} \omega_j(\bar{\mathbf{k}}, q) \cdot v_{\bar{\mathbf{k}}+\mathbf{e}_j}^h & l = n - j + 1; \\ 0 & l \neq n - j + 1; \end{cases} \\ T_{\bar{\mathbf{k}},j}^-(\psi_{j,l}^-) &= \begin{cases} \varpi_j(\bar{\mathbf{k}}, q) \cdot v_{\bar{\mathbf{k}}-\mathbf{e}_j}^h & l = n - j + 1; \\ 0 & l \neq n - j + 1. \end{cases} \end{aligned}$$

Here $v_{\bar{\mathbf{k}}}^h$, $\varpi_j(\bar{\mathbf{k}}, q)$ and $\omega_j(\bar{\mathbf{k}}, q)$ are introduced in Propositions 4, 7, and 10.

Lemma 4 *For all $\xi \in \mathfrak{p}_q^- \oplus \mathfrak{p}_q^+$, $u \in V_{\bar{\mathbf{k}}}$, $j = 1, \dots, n$ the following holds:*

$$\begin{aligned} P_{\bar{\mathbf{k}}+\mathbf{e}_j}(\pi_{\alpha,\beta}(\xi)u) &= q^{-\beta-n/2}[\beta - k_j + j - 1]_q T_{\bar{\mathbf{k}},j}^+(\xi \otimes u); \\ P_{\bar{\mathbf{k}}-\mathbf{e}_j}(\pi_{\alpha,\beta}(\xi)u) &= q^{\alpha+n/2}[\alpha + k_j + n - j]_q T_{\bar{\mathbf{k}},j}^-(\xi \otimes u). \end{aligned}$$

Proof. The proof completely repeats the proof of Lemma 9.10 of the paper [8]. \square

Using the last lemma, we can rewrite the $U_q \mathfrak{su}_{n,n}$ -invariance condition of the scalar product as follows: for all $\xi \in \mathfrak{p}_q^- \oplus \mathfrak{p}_q^+$, $u \in V_{\bar{\mathbf{k}}}$, $v \in V_{\bar{\mathbf{k}}+\mathbf{e}_j}$, $j = 1, \dots, n$

$$\begin{aligned} q^{-\beta-n/2}[\beta - k_j + j - 1]_q c_{\bar{\mathbf{k}}+\mathbf{e}_j} &< T_{\bar{\mathbf{k}},j}^+(\xi \otimes u), v >_{\bar{\mathbf{k}}+\mathbf{e}_j} = \\ &= \overline{q^{\alpha+n/2}[\alpha + (k_j + 1) + n - j]_q c_{\bar{\mathbf{k}}}} < u, T_{\bar{\mathbf{k}}+\mathbf{e}_j,j}^-(\xi^* \otimes v) >_{\bar{\mathbf{k}}} . \end{aligned}$$

Proposition 14 $< T_{\bar{\mathbf{k}},j}^+(\xi \otimes u), v >_{\bar{\mathbf{k}}+\mathbf{e}_j} = - < u, T_{\bar{\mathbf{k}}+\mathbf{e}_j,j}^-(\xi^* \otimes v) >_{\bar{\mathbf{k}}}$ for all $j = 1, \dots, n$.

Proof. Since the maps $T_{\bar{\mathbf{k}},j}^\pm$ does not depend on α , $\beta \in \mathcal{D}$, it is enough to consider only the special case $\text{Re}(\alpha + \beta) = -n$. In this case the representation $\pi_{\alpha,\beta}$ is unitary, thus we can put $c_{\bar{\mathbf{k}}} = 1$ for all $\bar{\mathbf{k}} \in \widehat{K}$. Since $\overline{q^{\alpha+n}} = q^{-\beta}$, we see that

$$q^{-\beta-n/2} \frac{q^{\beta-k_j+j-1} - q^{-\beta+k_j-j+1}}{q - q^{-1}} (< T_{\bar{\mathbf{k}},j}^+(\xi \otimes u), v >_{\bar{\mathbf{k}}+\mathbf{e}_j} + < u, T_{\bar{\mathbf{k}}+\mathbf{e}_j,j}^-(\xi^* \otimes v) >_{\bar{\mathbf{k}}}) = 0.$$

If we consider non-integral α , β , then $q^{\beta-k_j+j-1} - q^{-\beta+k_j-j+1}$ does not vanish. This completes the proof. \square

Recall that $\alpha, \beta \notin \mathbb{Z}$. Thus the $U_q \mathfrak{su}_{n,n}$ -invariance condition of the scalar product can be rewritten as follows: for all $\bar{\mathbf{k}} \in \widehat{K}$, $j = 1, \dots, n$

$$(1 - q^{2(-\beta+k_j+1-j)}) \overline{(1 - q^{2(\alpha+(k_j+1)+n-j)})}^{-1} = - \frac{c_{\bar{\mathbf{k}}}}{c_{\bar{\mathbf{k}}+\mathbf{e}_j}}. \quad (18)$$

Since the scalar product must be positive definite, we have the following necessary conditions for the unitarizability of modules of the degenerate principal series (recall that $q = e^{-h/2}$): for all $\bar{\mathbf{k}} \in \widehat{K}$, $j = 1, \dots, n$

$$\text{sh} \frac{h}{2}(\beta - k_j + j - 1) \overline{(\text{sh} \frac{h}{2}(\alpha + (k_j + 1) + n - j))}^{-1} < 0.$$

Using these inequalities, we can present the following series of simple unitary *representations of degenerate principal series* related to the Shilov boundary.

The principal unitary series: $\text{Re}(\alpha + \beta) = -n$, $\alpha, \beta \notin \mathbb{Z}$. In this case all representations are unitary. The invariant scalar product provided by the $U_q\mathfrak{su}_{n,n}$ -invariant integral [20].

The complementary series: $\text{Im}(\alpha + \beta) = 0$, $|\text{Re}\alpha + n| < 1$, $|\text{Re}\beta| < 1$, $(\text{Re}\alpha + n)\text{Re}\beta < 0$, $\alpha, \beta \notin \mathbb{Z}$. In this case the representations $\pi_{\alpha,\beta}$ are unitary too. (The required invariant scalar product (\cdot, \cdot) is defined by the coefficients $\{c_{\overline{\mathbf{k}}}\}$ as follows: let $c_{\overline{\mathbf{0}}} = 1$, other coefficients are computed from recurrent relations such as (18).)

The strange series: $\text{Im}\alpha = \frac{\pi}{h}$. For such values of the parameters the respective representations $\pi_{\alpha,\beta}$ are irreducible and unitary. This series of representations has no classical analogue. For the first time it appears in unpublished works of L.Korogodsky and in A.Klimyk and V.Groza's paper (see [6]).

Now let $\alpha, \beta \in \mathbb{Z}$. (Recall that in this case $\pi_{\alpha,\beta}$ is reducible.) For such α, β there might exist unitarizable simple submodules in the respective module (we will mention them below), although the module is not unitarizable. For each simple submodule the same arguments as in "general case" on the $U_q\mathfrak{su}_{n,n}$ -invariance of scalar product can be applied. In each case we have the necessary conditions like (18), however they must be satisfied only on a certain part of \widehat{K} . Consider all possible cases:

Case 1. $\alpha + \beta \geq 2 - n$. In this case the representation is not unitary and its unique irreducible subrepresentation is not unitary too.

Case 2. $\alpha + \beta = 1 - n$. In this case there exist n irreducible unitary subrepresentations of the representation $\pi_{\alpha, 1-n-\alpha}$. Precisely, V_j^s (see (16)) is a simple submodule in V for any $j = 1, \dots, n$. Notice that each V_j^s can be equipped with a $U_q\mathfrak{su}_{n,n}$ -invariant scalar product (\cdot, \cdot) . Such modules are called small representations because they have "poor" decompositions into isotypic components.

Case 3. $\alpha + \beta = -n$. In this case the representations are completely reducible, their irreducible subrepresentations V_i^s , $i = 1, \dots, n+1$ (see Section V) are unitary (actually, the required invariant scalar product is the same as for the principal unitary series).

Case 4. $\alpha + \beta \leq -1 - n$. In this case the submodules V_i^s , $i = 1, \dots, n+1$ are unitary although there exist non-unitarizable quotient modules in V .

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IX Appendix

Let us prove Lemma 2. This proof is a q -analogue of the proof of Lemma 3.4 from [8].

Proof of Lemma 2. Statements (9)-(12) can be easily checked.

For example, check the equality $K_j F_{mj} = q F_{mj} K_j$. For $j - m = 1$, i.e., $m = j - 1$, we see that $K_j F_{j-1,j} = K_j F_{j-1} K_{j-1} = F_{j-1} K_{j-1} K_j = q F_{j-1,j} K_j$. Assume that for $j - m < r$

equations (9)-(12) are proved. Let $j - m = r$. Then,

$$\begin{aligned}
K_j F_{mj} &= K_j (F_{m+1,j} F_m K_m + \sum_{s=m+2}^j (-1)^{s+m+1} F_{sj} \operatorname{ad}_{F_{s-1}} \dots \operatorname{ad}_{F_{m+1}} (F_m K_m) K(j, m+1, s-1)) \\
&= K_j F_{m+1,j} F_m K_m + \sum_{s=m+2}^j (-1)^{s+m+1} K_j F_{sj} \operatorname{ad}_{F_{s-1}} \dots \operatorname{ad}_{F_{m+1}} (F_m K_m) K(j, m+1, s-1) \\
&= q F_{m+1,j} K_j F_m K_m + q \sum_{s=m+2}^{j-1} (-1)^{s+m+1} F_{sj} K_j \operatorname{ad}_{F_{s-1}} \dots \operatorname{ad}_{F_{m+1}} (F_m K_m) K(j, m+1, s-1) \\
&\quad + q (-1)^{j+m+1} \operatorname{ad}_{F_{j-1}} \dots \operatorname{ad}_{F_{m+1}} (F_m K_m) K_j = q F_{mj} K_j.
\end{aligned}$$

The proof is completed by induction.

Using (12), prove equality (14). Recall that $[x]_q = \frac{q^x - q^{-x}}{q - q^{-1}}$. In the next equality we assume that $Z = E_m$: $E_m F_{mj} =$

$$\begin{aligned}
&= \sum_{s=m+2}^j (-1)^{s+m+1} E_m F_{sj} \operatorname{ad}_{F_{s-1}} \dots \operatorname{ad}_{F_{m+1}} (F_m K_m) K(j, m+1, s-1) + E_m F_{m+1,j} F_m K_m \\
&\equiv \sum_{s=m+2}^j (-1)^{s+m+1} F_{sj} E_m \operatorname{ad}_{F_{s-1}} \dots \operatorname{ad}_{F_{m+1}} (F_m K_m) K(j, m+1, s-1) + q F_{m+1,j} E_m F_m K_m \\
&\equiv q \sum_{s=m+2}^j (-1)^{s+m} F_{sj} \operatorname{ad}_{F_{s-1}} \dots \operatorname{ad}_{F_{m+2}} (F_{m+1} K_{m+1}) K_m^2 K(j, m+1, s-1) + q F_{m+1,j} [H_m]_q K_m \\
&\quad = q \sum_{s=m+2}^j (-1)^{s+m+2} F_{sj} \operatorname{ad}_{F_{s-1}} \dots \operatorname{ad}_{F_{m+2}} (F_{m+1} K_{m+1}) K(j, m+2, s-1) K_m^2 \\
&\quad \cdot q^{j-m-1} K_{m+1} \dots K_{j-1} [H_{m+1} + \dots + H_{j-1} + j - m - 1]_q + q F_{m+1,j} [H_m]_q K_m \\
&= q F_{m+1,j} (q^{j-m-1} K_m^2 K_{m+1} \dots K_{j-1} [H_{m+1} + \dots + H_{j-1} + j - m - 1]_q + [H_m]_q K_m) \\
&\quad = F_{m+1,j} q^{j-m} K_m \dots K_{j-1} [H_m + \dots + H_{j-1} + j - m - 1]_q \pmod{U_q \mathfrak{sl}_n \cdot E_m}.
\end{aligned}$$

Prove equality (13) by induction. If $j - m = 2$ and $m < i < j$, then $i = j - 1$, and (13) means that $E_{j-1} F_{j-2,j} \equiv 0 \pmod{U_q \mathfrak{sl}_n \cdot E_{j-1}}$. It can be proved as follows:

$$\begin{aligned}
E_{j-1} F_{j-2,j} &= E_{j-1} (F_{j-1,j} F_{j-2} K_{j-2} K(j, j-1, j-2) - \operatorname{ad}_{F_{j-1}} (F_{j-2} K_{j-2}) K(j, j-1, j-1)) \\
&\equiv [H_{j-1}]_q K_{j-1} F_{j-2} K_{j-2} - q F_{j-2} K_{j-2} K_{j-1} [H_{j-1} + 1]_q = 0 \pmod{U_q \mathfrak{sl}_n \cdot E_{j-1}}.
\end{aligned}$$

For the inductive step it is sufficient to check that for all $m < i < j$

$$\begin{aligned}
E_i F_{mj} &= \sum_{s=m+2}^j (-1)^{s+m+1} E_i F_{sj} \operatorname{ad}_{F_{s-1}} \dots \operatorname{ad}_{F_{m+1}} (F_m K_m) K(j, m+1, s-1) + E_i F_{m+1,j} F_m K_m \\
&\equiv \sum_{s=m+2}^{i-1} (-1)^{s+m+1} E_i F_{sj} \operatorname{ad}_{F_{s-1}} \dots \operatorname{ad}_{F_{m+1}} (F_m K_m) K(j, m+1, s-1) + E_i F_{m+1,j} F_m K_m \\
&\quad + \sum_{s=i+1}^j (-1)^{s+m+1} E_i F_{sj} \operatorname{ad}_{F_{s-1}} \dots \operatorname{ad}_{F_{m+1}} (F_m K_m) K(j, m+1, s-1) \\
&\quad + (-1)^{i+m+1} E_i F_{ij} \operatorname{ad}_{F_{i-1}} \dots \operatorname{ad}_{F_{m+1}} (F_m K_m) K(j, m+1, i-1),
\end{aligned}$$

(we use (12) and (14) and assume $Z = E_i$). By the inductive hypothesis, for $s < i$ we have $E_i F_{sj} \equiv 0 \pmod{U_q \mathfrak{sl}_n \cdot E_i}$, therefore for all $m < s < i$ there exists an element $X_s \in U_q \mathfrak{sl}_n$ such that $E_i F_{sj} = X_s E_i$. From (12), $E_i F_{sj} = F_{sj} E_i$. From (14), $E_i F_{ij} = q F_{i+1,j} q^{j-i-1} K_i \dots K_{j-1} [H_i + \dots + H_{j-1} + j - i - 1]_q$. Thus,

$$\begin{aligned}
E_i F_{mj} &\equiv \sum_{s=m+2}^{i-1} (-1)^{s+m+1} X_s E_i \operatorname{ad}_{F_{s-1}} \dots \operatorname{ad}_{F_{m+1}} (F_m K_m) K(j, m+1, s-1) \\
&\quad + X_{m+1} E_i F_m K_m + \sum_{s=i+1}^j (-1)^{s+m+1} F_{sj} E_i \operatorname{ad}_{F_{s-1}} \dots \operatorname{ad}_{F_{m+1}} (F_m K_m) K(j, m+1, s-1) \\
&\quad + (-1)^{i+m+1} F_{i+1,j} q^{j-i-1} K_i \dots K_{j-1} [H_i + \dots + H_{j-1} + j - i - 1]_q \operatorname{ad}_{F_{i-1}} \dots \operatorname{ad}_{F_{m+1}} (F_m K_m) \\
&\quad \cdot K(j, m+1, i-1) \equiv (-1)^{i+m+1} q F_{i+1,j} \operatorname{ad}_{F_{i-1}} \dots \operatorname{ad}_{F_{m+1}} (F_m K_m) K(j, m+1, i) \\
&\quad + (-1)^{i+m} F_{i+1,j} E_i \operatorname{ad}_{F_i} \dots \operatorname{ad}_{F_{m+1}} (F_m K_m) K(j, m+1, i) = 0 \pmod{U_q \mathfrak{sl}_n \cdot E_i}. \quad \square
\end{aligned}$$

The proof of Lemma 3 is similar.

Let us prove Proposition 7. We just have to compute the coefficients $c_j(\beta, k_j)$. Recall that there is a $U_q \mathfrak{sl}_n \otimes U_q \mathfrak{sl}_n$ -isomorphism $j_1 : \mathfrak{p}_q^+ \simeq \mathbb{C}^n \otimes (\mathbb{C}^n)^*$, where \mathbb{C}^n is the vector representation of $U_q \mathfrak{sl}_n$. The isomorphism j_1^{-1} on the elements of the standard basis for $\mathbb{C}^n \otimes (\mathbb{C}^n)^*$ is defined as follows:

$$\begin{aligned}
&j_1^{-1} \begin{pmatrix} \varepsilon_1 \otimes \varepsilon_1^* & \dots & \varepsilon_1 \otimes \varepsilon_n^* \\ \dots & \dots & \dots \\ \varepsilon_{n-1} \otimes \varepsilon_1^* & \dots & \dots \\ \varepsilon_n \otimes \varepsilon_1^* & \dots & \varepsilon_n \otimes \varepsilon_n^* \end{pmatrix} = \\
&= \begin{pmatrix} \operatorname{ad}_{E_1} \dots \operatorname{ad}_{E_{n-1}} E_n & \dots & \dots & (-1)^{n-1} \operatorname{ad}_{E_{2n-1}} \dots \operatorname{ad}_{E_{n+1}} \operatorname{ad}_{E_1} \dots \operatorname{ad}_{E_{n-1}} E_n \\ \dots & \dots & \dots & \dots \\ \operatorname{ad}_{E_{n-1}} E_n & \dots & \dots & \dots \\ E_n & -\operatorname{ad}_{E_{n+1}} E_n & \dots & (-1)^{n-1} \operatorname{ad}_{E_{2n-1}} \dots \operatorname{ad}_{E_{n+1}} E_n \end{pmatrix}
\end{aligned}$$

(This follows from the equalities $\operatorname{ad}_{F_j} E_n = 0$, $\operatorname{ad}_{E_j}^2 E_n = 0$ for $j = 1, \dots, 2n-1, j \neq n$, $\operatorname{ad}_{K_j} E_n = E_n$ for $j = 1, \dots, n-2, n+2, \dots, 2n-1$, $\operatorname{ad}_{K_j} E_n = q^{-1} E_n$ for $j = n-1$ or $j = n+1$.) Consider the following embeddings of vector spaces

$$\begin{aligned}
\iota_1 : U_q \mathfrak{sl}_n &\hookrightarrow U_q \mathfrak{sl}_n \otimes U_q \mathfrak{sl}_n, & \xi &\mapsto \xi \otimes 1; \\
\iota_2 : U_q \mathfrak{sl}_n &\hookrightarrow U_q \mathfrak{sl}_n \otimes U_q \mathfrak{sl}_n, & \xi &\mapsto 1 \otimes \xi.
\end{aligned}$$

Set $\xi^{(1)} = \iota_1(\xi)$ and $\xi^{(2)} = \iota_2(\xi)$.

From Propositions 5 and 6, we deduce that for all $j = 1, \dots, n, \bar{\mathbf{k}} \in \widehat{K}$

$$\begin{aligned} \mathcal{M}_{\bar{\mathbf{k}}}^+(\zeta_j \otimes \xi_{n-j+1}) &= \mathcal{M}_{\bar{\mathbf{k}}}^+\left(\sum_{m=1}^j (-q^2)^{m-1} \varepsilon_m \otimes F_{mj}^{(1)} K_-^{(1)}(j, 1, m-1) u\right. \\ &\otimes \sum_{m=n-j+1}^n \varepsilon_m^* \otimes S_{n-j+1,m}^{(2)} L_-^{(2)}(n-j+1, m+1, n) u^*) = \mathcal{M}_{\bar{\mathbf{k}}}^+\left(\sum_{m=1}^j \sum_{l=n-j+1}^n (-q^2)^{m-1} \varepsilon_m\right. \\ &\left. \otimes \varepsilon_l^* \otimes F_{mj} K_-(j, 1, m-1) u \otimes S_{n-j+1,l} L_-(n-j+1, l+1, n) u^*\right). \end{aligned}$$

Proposition 15 For all $j = 1, \dots, n, \bar{\mathbf{k}} \in \widehat{K}$

$$\mathcal{M}_{\bar{\mathbf{k}}}^+(\zeta_j \otimes \xi_{n-j+1}) = \lambda^-(n-j+1, n-j+2, n) \mathcal{M}_{\bar{\mathbf{k}}}^+(\zeta_j \otimes \varepsilon_{n-j+1}^* \otimes u^*),$$

where $L_-^{(2)}(n-j+1, n-j+2, n)(v_{\bar{\mathbf{k}}}^h) = \lambda^-(n-j+1, n-j+2, n) v_{\bar{\mathbf{k}}}^h$.

Proof. In the same way as in [8], we have

$$\begin{aligned} \mathcal{M}_{\bar{\mathbf{k}}}^+(\zeta_j \otimes \xi_{n-j+1}) &= \mathcal{M}_{\bar{\mathbf{k}}}^+\left(\sum_{m=1}^j \sum_{l=n-j+1}^n (-q^2)^{m-1} \varepsilon_m \otimes \varepsilon_l^* \otimes F_{mj}^{(1)} K_-^{(1)}(j, 1, m-1) u\right. \\ &\left. \otimes S_{n-j+1,l}^{(2)} L_-^{(2)}(n-j+1, l+1, n) u^*\right) = \mathcal{M}_{\bar{\mathbf{k}}}^+\left(\sum_{m=1}^j (-q^2)^{m-1} \varepsilon_m \otimes \varepsilon_{n-j+1}^* \otimes F_{mj}^{(1)} K_-^{(1)}(j, 1, m-1) u\right. \\ &\left. \otimes L_-^{(2)}(n-j+1, n-j+2+1, n) u^*\right) + \mathcal{M}_{\bar{\mathbf{k}}}^+\left(\sum_{m=1}^j \sum_{l=n-j+2}^n (-q^2)^{m-1} \varepsilon_m\right. \\ &\left. \otimes \varepsilon_l^* \otimes F_{mj}^{(1)} K_-^{(1)}(j, 1, m-1) u \otimes S_{n-j+1,l}^{(2)} L_-^{(2)}(n-j+1, l+1, n) u^*\right) \\ &= \lambda^-(n-j+1, n-j+2, n) \mathcal{M}_{\bar{\mathbf{k}}}^+\left(\sum_{m=1}^j (-q^2)^{m-1} \varepsilon_m \otimes \varepsilon_{n-j+1}^* \otimes F_{mj}^{(1)} K_-^{(1)}(j, 1, m-1) u \otimes u^*\right) \\ &+ \mathcal{M}_{\bar{\mathbf{k}}}^+\left(\sum_{m=1}^j \sum_{l=n-j+2}^n (-q^2)^{m-1} \varepsilon_m \otimes \varepsilon_l^* \otimes F_{mj}^{(1)} K_-^{(1)}(j, 1, m-1) u \otimes S_{n-j+1,l}^{(2)} L_-^{(2)}(n-j+1, l+1, n) u^*\right) \\ &= \lambda^-(n-j+1, n-j+2, n) \mathcal{M}_{\bar{\mathbf{k}}}^+(\zeta_j \otimes \varepsilon_{n-j+1}^* \otimes u^*) \\ &+ \mathcal{M}_{\bar{\mathbf{k}}}^+\left(\sum_{l=n-j+2}^n \sum_{m=1}^j (-q^2)^{m-1} \varepsilon_m \otimes (\text{ad}_{E_{n-l-2}} \dots \text{ad}_{E_j})^{(2)} \varepsilon_{n-j+1}^*\right. \\ &\left. \otimes F_{mj}^{(1)} K_-^{(1)}(j, 1, m-1) u \otimes S_{n-j+1,l}^{(2)} L_-^{(2)}(n-j+1, l+1, n) u^*\right) \\ &= \lambda^-(n-j+1, n-j+2, n) \mathcal{M}_{\bar{\mathbf{k}}}^+(\zeta_j \otimes \varepsilon_{n-j+1}^* \otimes u^*) \\ &+ \sum_{l=n-j+1}^n (\text{ad}_{E_{n-l-2}} \dots \text{ad}_{E_j})^{(2)} S_{n-j+1,l}^{(2)} L_-^{(2)}(n-j+1, l+1, n) \mathcal{M}_{\bar{\mathbf{k}}}^+(\zeta_j \otimes \varepsilon_{n-j+1}^* \otimes u^*). \end{aligned}$$

The vector $\mathcal{M}^+_{\bar{\mathbf{k}}}(\zeta_j \otimes \varepsilon_{n-j+1}^* \otimes u^*) \in V_{\bar{\mathbf{k}}+\mathbf{e}_j}$ and is a $U_q \mathfrak{sl}_n \otimes 1$ -highest vector. Therefore $\mathcal{M}^+_{\bar{\mathbf{k}}}(\zeta_j \otimes \varepsilon_{n-j+1}^* \otimes u^*) \in V_{\bar{\mathbf{k}}+\mathbf{e}_j}$, $\mathcal{M}^+_{\bar{\mathbf{k}}}(\zeta_j \otimes \varepsilon_{n-j+1}^* \otimes u^*) = c \cdot v_{\bar{\mathbf{k}}+\mathbf{e}_j}^h$ with some $c \in \mathbb{C}$. Now we conclude that in the obtained expression all summands except the first equal 0. \square

To find $c_j(\beta, k_j)$ we must compute

$$\begin{aligned} \mathcal{M}^+_{\bar{\mathbf{k}}}(\zeta_j \otimes \varepsilon_{n-j+1}^* \otimes u^*) &= \mathcal{M}^+_{\bar{\mathbf{k}}} \left(\sum_{m=1}^j (-q^2)^{m-1} \varepsilon_m \otimes \varepsilon_{n-j+1}^* \otimes F_{mj}^{(1)} K_-^{(1)}(j, 1, m-1) u \otimes u^* \right) \\ &= \sum_{m=1}^j (-q^2)^{m-1} \pi_{\alpha, \beta}((-1)^{j-1} \text{ad}_{E_{n+j-1}} \dots \text{ad}_{E_{n+1}} \text{ad}_{E_m} \dots \text{ad}_{E_{n-1}} E_n) \\ &\quad F_{mj}^{(1)} K_-^{(1)}(j, 1, m-1) (v_{\bar{\mathbf{k}}}^h). \quad (\text{A1}) \end{aligned}$$

We need some auxiliary lemmas. Recall that in this paper we introduce the notation for q -minors of the matrix \mathbf{z} (see (4)). Set $\mathbf{z}_{a_1, \dots, a_k}^{\wedge k} = \mathbf{z}_{\{a_1, \dots, a_k\}}^{\wedge k \{1, \dots, k\}}$.

Lemma 5 For all $1 \leq m \leq k \leq j-2$

$$\begin{aligned} &(-q)^{j-k-1} \mathbf{z}^{\wedge j-1} \mathbf{z}_{1, \dots, m-1, m+1, \dots, j}^{\wedge k} - \sum_{s=k+1}^{j-2} (-q)^{s-k-1} \mathbf{z}_{1, \dots, s-1, s+1, \dots, j}^{\wedge k} \mathbf{z}_{1, \dots, m-1, m+1, \dots, s}^{\wedge k} \\ &= \mathbf{z}_{1, \dots, m-1, m+1, \dots, j}^{\wedge j-1} \mathbf{z}^{\wedge k}. \quad \square \end{aligned}$$

Lemma 6 For all $1 \leq m \leq j \leq n$ we have $F_{mj} = q^{j-m-1} G_{mj}$, where

$$G_{mj} = F_m K_m F_{m+1, j} + \sum_{s=m+2}^j (-q)^{s-m-1} \text{ad}_{F_{s-1}} \dots \text{ad}_{F_{m+1}} (F_m K_m) F_{sj} K_-(j, m+1, s-1). \square$$

Lemma 7 For all $1 \leq m \leq j \leq n$

$$G_{mj}(v_{\bar{\mathbf{k}}}^h) = (q^{1/2})^{j-m} \kappa_-(j, m, j-1) z_{1, \dots, m-1, m+1, \dots, j}^{\wedge j-1} \frac{v_{\bar{\mathbf{k}}}^h}{z^{\wedge j-1}},$$

where $K_-(j, m+1, j-1)(v_{\bar{\mathbf{k}}}^h) = \kappa_-(j, m, j-1) v_{\bar{\mathbf{k}}}^h$.

Proof. We prove this lemma by induction. For $j-m=1$ the statement is obvious, since

$$\begin{aligned} G_{m, m+1}(v_{\bar{\mathbf{k}}}^h) &= F_m K_m ((z^{\wedge 1})^{k_1-k_2} \dots (z^{\wedge n})^{k_n}) \\ &= q^{k_m-k_{m+1}} q^{1/2} [k_m - k_{m+1}]_q (z^{\wedge 1})^{k_1-k_2} \dots (z^{\wedge m-1})^{k_{m-1}-k_m} z_{1, \dots, m-1, m+1}^{\wedge m} (z^{\wedge m})^{k_m-k_{m+1}-1} \\ &\quad \cdot (z^{\wedge m+1})^{k_{m+1}-k_{m+2}} \dots (z^{\wedge n})^{k_n} = q^{1/2} \kappa_-(m+1, m, m) z_{1, \dots, m-1, m+1}^{\wedge m} v_{\bar{\mathbf{k}}-\mathbf{e}_m}^h. \end{aligned}$$

For the proof of the inductive step we use two previous lemmas. By Lemma 6, we have

$$\begin{aligned} G_{mj}(v_{\bar{\mathbf{k}}}^h) &= \sum_{s=m+2}^j (-q)^{s-m-1} \text{ad}_{F_{s-1}} \dots \text{ad}_{F_{m+1}} (F_m K_m) \cdot F_{sj} K_-(j, m+1, s-1) (v_{\bar{\mathbf{k}}}^h) \\ &\quad + F_m K_m F_{m+1, j} (v_{\bar{\mathbf{k}}}^h) = F_m K_m F_{m+1, j} (v_{\bar{\mathbf{k}}}^h) \\ &\quad + \sum_{s=m+2}^j (-q)^{s-m-1} \kappa_-(j, m+1, s-1) \text{ad}_{F_{s-1}} \dots \text{ad}_{F_{m+1}} (F_m K_m) F_{sj} (v_{\bar{\mathbf{k}}}^h). \end{aligned}$$

By the inductive hypothesis, for all $j - s < j - m$

$$\begin{aligned}
G_{mj}(v_{\mathbf{k}}^h) &= q^{1/2(j-m-1)} \kappa_{-}(j, m+1, j-1) F_m K_m(z_{1,\dots,m-1,m+1,\dots,j}^{\wedge j-1} \frac{v_{\mathbf{k}}^h}{z^{\wedge j-1}}) + \sum_{s=m+2}^j (-q)^{s-m-1} \\
&\cdot q^{1/2(j-s)} \kappa_{-}(j, m+1, s-1) \kappa_{-}(j, s, j-1) \text{ad}_{F_{s-1}} \dots \text{ad}_{F_{m+1}}(F_m K_m)(z_{1,\dots,m-1,m+1,\dots,j}^{\wedge j-1} \frac{v_{\mathbf{k}}^h}{z^{\wedge j-1}}) \\
&= q^{1/2(j-m-1)} \kappa_{-}(j, m+1, j-1) \cdot (F_m K_m(z_{1,\dots,m-1,m+1,\dots,j}^{\wedge j-1} \frac{v_{\mathbf{k}}^h}{z^{\wedge j-1}}) \\
&\quad + \sum_{s=m+2}^j (-q^{1/2})^{s-m-1} \text{ad}_{F_{s-1}} \dots \text{ad}_{F_{m+1}}(F_m K_m)(z_{1,\dots,s-1,s+1,\dots,j}^{\wedge j-1} \frac{v_{\mathbf{k}}^h}{z^{\wedge j-1}})).
\end{aligned}$$

Using the explicit formulas for the $U_q \mathfrak{sl}_{2n}$ -action in $\mathbb{C}[\text{Mat}_n]_q$ and properties of the comultiplication (see Section II), we obtain that

$$\begin{aligned}
&\text{ad}_{F_{s-1}} \dots \text{ad}_{F_{m+1}}(F_m K_m)(z_{1,\dots,s-1,s+1,\dots,j}^{\wedge j-1} \frac{v_{\mathbf{k}}^h}{z^{\wedge j-1}}) \\
&= (q^{1/2})^{s-m} (q^{k_m-k_{m+1}} [k_m - k_{m+1}]_q (z^{\wedge 1})^{k_1-k_2} \dots (z^{\wedge m-1})^{k_{m-1}-k_m} z_{1,\dots,m-1,m+1,\dots,s}^{\wedge m} \\
&\quad \cdot (z^{\wedge m})^{k_m-k_{m+1}-1} \dots (z^{\wedge s-1})^{k_{s-1}-k_s} z_{1,\dots,s-1,s+1,\dots,j}^{\wedge j-1} (z^{\wedge s})^{k_s-k_{s+1}} \dots (z^{\wedge n})^{k_n} \\
&\quad + (-q) q^{k_m-k_{m+1}+k_{m+1}-k_{m+2}} [k_{m+1} - k_{m+2}]_q (z^{\wedge 1})^{k_1-k_2} \dots (z^{\wedge m})^{k_m-k_{m+1}} \\
&\quad \cdot z_{1,\dots,m-1,m+1,\dots,s}^{\wedge m+1} (z^{\wedge m+1})^{k_{m+1}-k_{m+2}-1} \dots (z^{\wedge s-1})^{k_{s-1}-k_s} z_{1,\dots,s-1,s+1,\dots,j}^{\wedge j-1} \\
&\quad \cdot (z^{\wedge s})^{k_s-k_{s+1}} \dots (z^{\wedge n})^{k_n} + \dots + (-q)^{s-2} q^{k_m-k_{m+1}+\dots+k_{s-2}-k_{s-1}} [k_{s-2} - k_{s-1}]_q \\
&\quad \cdot (z^{\wedge 1})^{k_1-k_2} \dots (z^{\wedge s-2})^{k_{s-2}-k_{s-1}} z_{1,\dots,m-1,m+1,\dots,s}^{\wedge s-1} \cdot (z^{\wedge s-1})^{k_{s-1}-k_s-1} z_{1,\dots,s-1,s+1,\dots,j}^{\wedge j-1} \\
&\quad \cdot (z^{\wedge s})^{k_s-k_{s+1}} \dots (z^{\wedge n})^{k_n} + (-q)^{s-1} q^{k_m-k_{m+1}+\dots+k_{s-1}-k_s} [k_{s-1} - k_s]_q (z^{\wedge 1})^{k_1-k_2} \dots \\
&\quad \cdot (z^{\wedge s-2})^{k_{s-2}-k_{s-1}} (z^{\wedge s-1})^{k_{s-1}-k_s} z_{1,\dots,m-1,m+1,\dots,j}^{\wedge j-1} (z^{\wedge s})^{k_s-k_{s+1}} \dots (z^{\wedge n})^{k_n}).
\end{aligned}$$

Finally, by Lemma 5, we see that

$$G_{mj}(v_{\mathbf{k}}^h) = (q^{1/2})^{j-m} \kappa_{-}(j, m+1, j-1) \cdot z_{1,\dots,m-1,m+1,\dots,j}^{\wedge j-1} \frac{v_{\mathbf{k}}^h}{z^{\wedge j-1}} \cdot \frac{q^{2(j-2+k_1-k_j)} - 1}{q - q^{-1}}. \quad \square$$

By the last lemma and ((A1)), we can compute the coefficients $c_j(\beta, k_j)$ introduced in Proposition 7.

Proposition 16 For all $1 \leq j \leq n$

$$\mathcal{M}_{\mathbf{k}}^+(\zeta_j \otimes \varepsilon_{n-j+1}^* \otimes u^*) = q^{-\beta-n/2+k_j+j} [\beta - k_j + j - 1]_q \kappa_{-}(j, 1, j-1) v_{\mathbf{k}+\mathbf{e}_j}^h.$$

Proof We have

$$\begin{aligned}
\mathcal{M}_{\mathbf{k}}^+(\zeta_j \otimes \varepsilon_{n-j+1}^* \otimes u^*) &= \sum_{m=1}^j (-q^2)^{m-1} \pi_{\alpha,\beta}((-1)^{j-1} \text{ad}_{E_{n+j-1}} \dots \text{ad}_{E_{n+1}} \text{ad}_{E_m} \dots \text{ad}_{E_{n-1}} E_n) \\
&\cdot F_{mj}^{(1)} K_{-}^{(1)}(j, 1, m-1) v_{\mathbf{k}}^h = \sum_{m=1}^j (-q^2)^{m-1} \kappa_{-}(j, 1, m-1) \\
&\cdot \pi_{\alpha,\beta}((-1)^{j-1} \text{ad}_{E_{n+j-1}} \dots \text{ad}_{E_{n+1}} \text{ad}_{E_m} \dots \text{ad}_{E_{n-1}} E_n) F_{mj}^{(1)} v_{\mathbf{k}}^h.
\end{aligned}$$

By Lemma 7,

$$\begin{aligned}
\mathcal{M}_{\bar{\mathbf{k}}}^+(\zeta_j \otimes \varepsilon_{n-j+1}^* \otimes u^*) &= \sum_{m=1}^{j-1} (-q^2)^{m-1} q^{j-m-1} (q^{1/2})^{j-m} \kappa_-(j, m, j-1) \kappa_-(j, 1, m-1) \\
&\quad \cdot (-1)^{j-1} \pi_{\alpha, \beta}(\text{ad}_{E_{n+j-1}} \dots \text{ad}_{E_{n+1}} \text{ad}_{E_m} \dots \text{ad}_{E_{n-1}} E_n) (z_{1, \dots, m-1, m+1, \dots, j}^{\wedge^{j-1}} \frac{v_{\bar{\mathbf{k}}}^h}{z^{\wedge^{j-1}}}) \\
&\quad + (-q^2)^{j-1} \kappa_-(j, 1, j-1) \pi_{\alpha, \beta}((-1)^{j-1} \text{ad}_{E_{n+j-1}} \dots \text{ad}_{E_{n+1}} \text{ad}_{E_j} \dots \text{ad}_{E_{n-1}} E_n) (v_{\bar{\mathbf{k}}}^h) \\
&= q^{3/2j-3} \kappa_-(j, 1, j-1) \sum_{m=1}^{j-1} (-1)^{j+m} q^{m/2} \\
&\quad \cdot \pi_{\alpha, \beta}(\text{ad}_{E_{n+j-1}} \dots \text{ad}_{E_{n+1}} \text{ad}_{E_m} \dots \text{ad}_{E_{n-1}} E_n) (z_{1, \dots, m-1, m+1, \dots, j}^{\wedge^{j-1}} \frac{v_{\bar{\mathbf{k}}}^h}{z^{\wedge^{j-1}}}) \\
&\quad + q^{2j-2} \kappa_-(j, 1, j-1) \pi_{\alpha, \beta}(\text{ad}_{E_{n+j-1}} \dots \text{ad}_{E_{n+1}} \text{ad}_{E_j} \dots \text{ad}_{E_{n-1}} E_n) (v_{\bar{\mathbf{k}}}^h).
\end{aligned}$$

In Section II the following morphism of $U_q \mathfrak{sl}_{2n}$ -modules was defined:

$$\iota : \mathbb{C}[\text{Mat}_n]_q \rightarrow \mathbb{C}[\text{Pl}_{n, 2n}]_{q, t}, \quad \iota(z^{\wedge k \{b_1, \dots, b_k\}}_{\{a_1, \dots, a_k\}}) = t^{-1} t^{\wedge n}_{\{1, \dots, n\}J},$$

with $J = \{n+1, \dots, 2n\} \setminus \{2n+1-b_1, \dots, 2n+1-b_k\} \cup \{a_1, \dots, a_k\}$. Therefore $\iota(z^{\wedge k}) = t^{-1} t^{\wedge n}_{\{1, \dots, n\}\{1, \dots, k, n+1, \dots, 2n-k\}}$. It follows that

$$\iota(v_{\bar{\mathbf{k}}}^h) = q^c t^{-k_1} (t^{\wedge n}_{\{1, \dots, n\}\{1, n+1, \dots, 2n-1\}})^{k_1-k_2} \dots (t^{\wedge n}_{\{1, \dots, n\}\{1, \dots, n\}})^{k_n},$$

where some $c \in \mathbb{C}$. Using the definition of $\pi_{\alpha, \beta}$, we obtain that for all $\xi \in U_q \mathfrak{sl}_{2n}$, $f \in \mathbb{C}[\text{Pl}_{n, 2n}]_{q, t}$

$$\pi_{\alpha, \beta}(\xi)(f) = q^c t^{-\beta} \xi \cdot (t^\beta f(t^{\wedge n}_{\{1, \dots, n\}\{1, \dots, n\}})^\alpha) (t^{\wedge n}_{\{1, \dots, n\}\{1, \dots, n\}})^{-\alpha}.$$

For $m < j$

$$\begin{aligned}
&(\text{ad}_{E_{n+j-1}} \dots \text{ad}_{E_{n+1}} \text{ad}_{E_m} \dots \text{ad}_{E_{n-1}} E_n) (t^{\wedge n \{1, \dots, n\}}_{\{1, \dots, m-1, m+1, \dots, j, n+1, \dots, 2n-j, 2n-m+1\}}) \\
&= (-1)^{m-j} (q^{-1/2})^{j+n-m} q^{m-j-1} t^{\wedge n}_{\{1, \dots, n\}\{1, \dots, j, n+1, \dots, 2n-j\}},
\end{aligned}$$

and for $m = j$ $(\text{ad}_{E_{n+j-1}} \dots \text{ad}_{E_{n+1}} \text{ad}_{E_j} \dots \text{ad}_{E_{n-1}} E_n) (t^{\wedge n}_{\{1, \dots, n\}\{1, \dots, j-1, n+1, \dots, 2n-j+1\}}) = (q^{-1/2})^n t^{\wedge n}_{\{1, \dots, n\}\{1, \dots, j, n+1, \dots, 2n-j\}}$. For other summands we use an analogue of Lemma 5. Finally, we have

$$\begin{aligned}
\mathcal{M}_{\bar{\mathbf{k}}}^+(\zeta_j \otimes \varepsilon_{n-j+1}^* \otimes u^*) &= \kappa_-(j, 1, j-1) q^{-n/2} \cdot (q^2 \sum_{m=1}^{j-1} q^{2m-2} + \frac{1 - q^{-2\beta+2k_1}}{1 - q^{-2}} \\
&+ \sum_{m=1}^{j-1} q^{-2\beta+2k_m} \frac{1 - q^{-2k_m+2k_{m+1}}}{1 - q^{-2}}) \cdot v_{\bar{\mathbf{k}}+\mathbf{e}_j}^h = q^{-\beta-n/2+k_j+j} \kappa_-(j, 1, j-1) [\beta - k_j + j - 1]_q v_{\bar{\mathbf{k}}+\mathbf{e}_j}^h.
\end{aligned}$$

□

Repeat the same arguments to prove Proposition 10. First, we have the explicit formulas for the isomorphism $j_2 : \mathfrak{p}_q^- \simeq (\mathbb{C}^n)^* \otimes \mathbb{C}^n$:

$$j_2^{-1} \begin{pmatrix} \varepsilon_1^* \otimes \varepsilon_1 & \dots & \varepsilon_n^* \otimes \varepsilon_1 \\ \dots & \dots & \dots \\ \varepsilon_1^* \otimes \varepsilon_n & \dots & \varepsilon_n^* \otimes \varepsilon_n \end{pmatrix} = \begin{pmatrix} (-1)^{n-1} \text{ad}_{F_1} \dots \text{ad}_{F_{n-1}}(K_n F_n) & \dots - \text{ad}_{F_{n-1}}(K_n F_n) & K_n F_n \\ \dots & \dots & \text{ad}_{F_{n+1}}(K_n F_n) \\ \dots & \dots & \dots \\ (-1)^{n-1} \text{ad}_{F_{2n-1}} \dots \text{ad}_{F_{n+1}} \text{ad}_{F_1} \dots \text{ad}_{F_{n-1}}(K_n F_n) & \dots & \text{ad}_{F_{2n-1}} \dots \text{ad}_{F_{n+1}}(K_n F_n) \end{pmatrix}$$

For the proof of Proposition 10 we must compute the following:

$$\begin{aligned} \mathcal{M}_{\bar{\mathbf{k}}}^-(\xi'_j \otimes \zeta'_{n-j+1}) &= \mathcal{M}_{\bar{\mathbf{k}}}^-\left(\sum_{m=j}^n \varepsilon_m^* \otimes S_{jm}^{(1)} L_-^{(1)}(j, m+1, n) u \otimes \sum_{m=1}^{n-j+1} (-q^2)^{m-1} \varepsilon_m \right. \\ &\quad \left. \otimes F_{m, n-j+1}^{(2)} K_-^{(2)}(n-j+1, 1, m-1) u^*\right) = \mathcal{M}_{\bar{\mathbf{k}}}^-\left(\sum_{m=j}^n \sum_{l=1}^{n-j+1} (-q^2)^{l-1} \varepsilon_m^* \otimes \varepsilon_l \right. \\ &\quad \left. \otimes S_{jm}^{(1)} L_-^{(1)}(j, m+1, n) u \otimes F_{l, n-j+1}^{(2)} K_-^{(2)}(n-j+1, 1, l-1) u^*\right). \end{aligned}$$

Proposition 17 *For all $1 \leq j \leq n$*

$$\mathcal{M}_{\bar{\mathbf{k}}}^-(\xi'_j \otimes \zeta'_{n-j+1}) = \lambda_-(j, j+1, n) \mathcal{M}_{\bar{\mathbf{k}}}^-(\varepsilon_j^* \otimes u \otimes \zeta'_{n-j+1}),$$

where $L_-^{(1)}(j, j+1, n) u = \lambda_-(j, j+1, n) u$.

The proof is similar to the proof of Proposition 15.

Therefore in order to find the coefficients $d_j(\alpha, k_j)$ introduced in Proposition 10 we must only compute

$$\begin{aligned} \mathcal{M}_{\bar{\mathbf{k}}}^-(\varepsilon_j^* \otimes u \otimes \zeta'_{n-j+1}) &= \mathcal{M}_{\bar{\mathbf{k}}}^-\left(\sum_{l=1}^{n-j+1} (-q^2)^{l-1} \varepsilon_j^* \otimes \varepsilon_l \otimes u \otimes F_{l, n-j+1} K_-(n-j+1, 1, l-1) u^*\right) \\ &= \sum_{l=1}^{n-j+1} (-q^2)^{l-1} \pi_{\alpha, \beta}(\text{ad}_{F_j} \dots \text{ad}_{F_{n-1}} \text{ad}_{F_{n-1+l}} \dots \text{ad}_{F_{n+1}}(K_n F_n)) \\ &\quad \cdot F_{l, n-j+1}^{(2)} K_-^{(2)}(n-j+1, 1, l-1) (v_{\bar{\mathbf{k}}}^h). \end{aligned}$$

These computations are analogous to the ones from the proof of Proposition 16.

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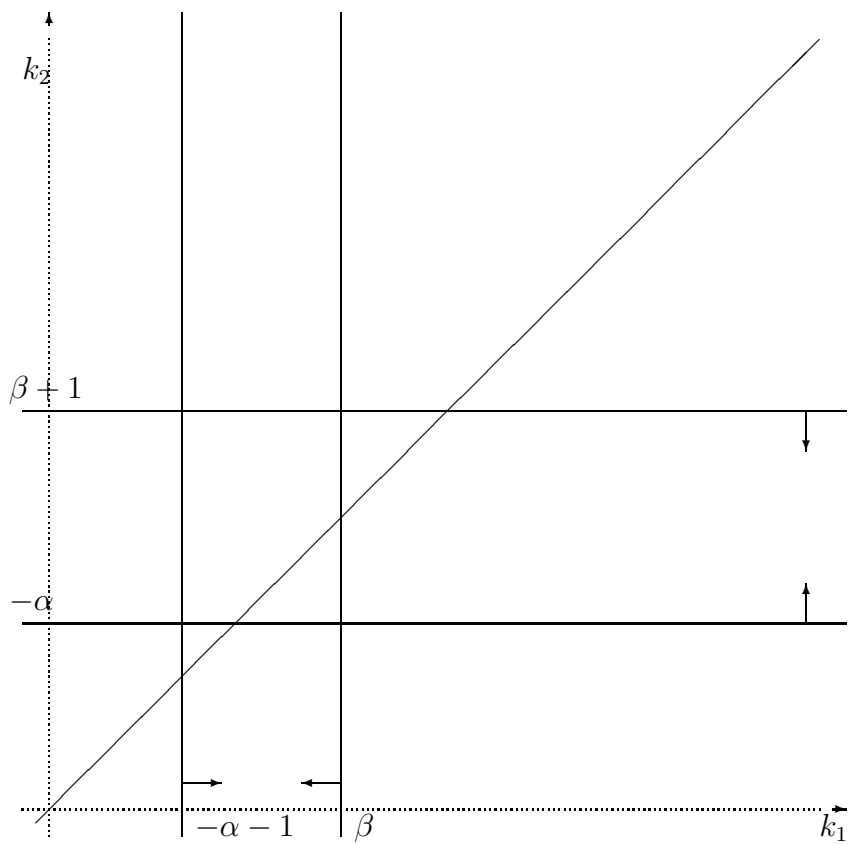


Fig.1. Structure of $\pi_{\alpha,\beta}$ with $\alpha + \beta \geq 0$.

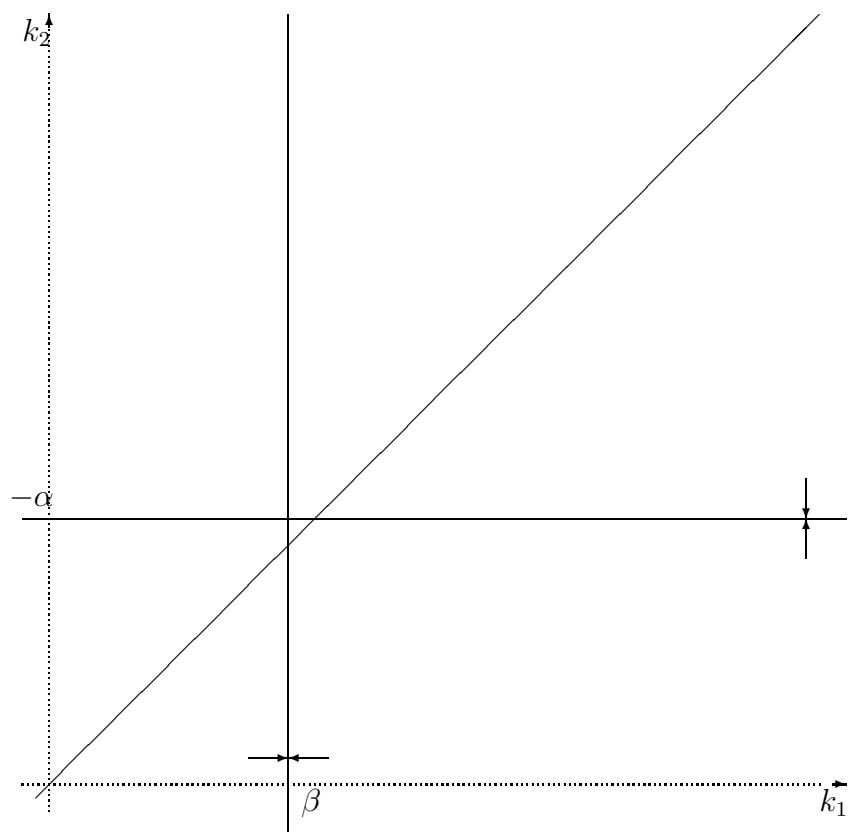


Fig.2. Structure of $\pi_{\alpha,\beta}$ with $\alpha + \beta = -1$.

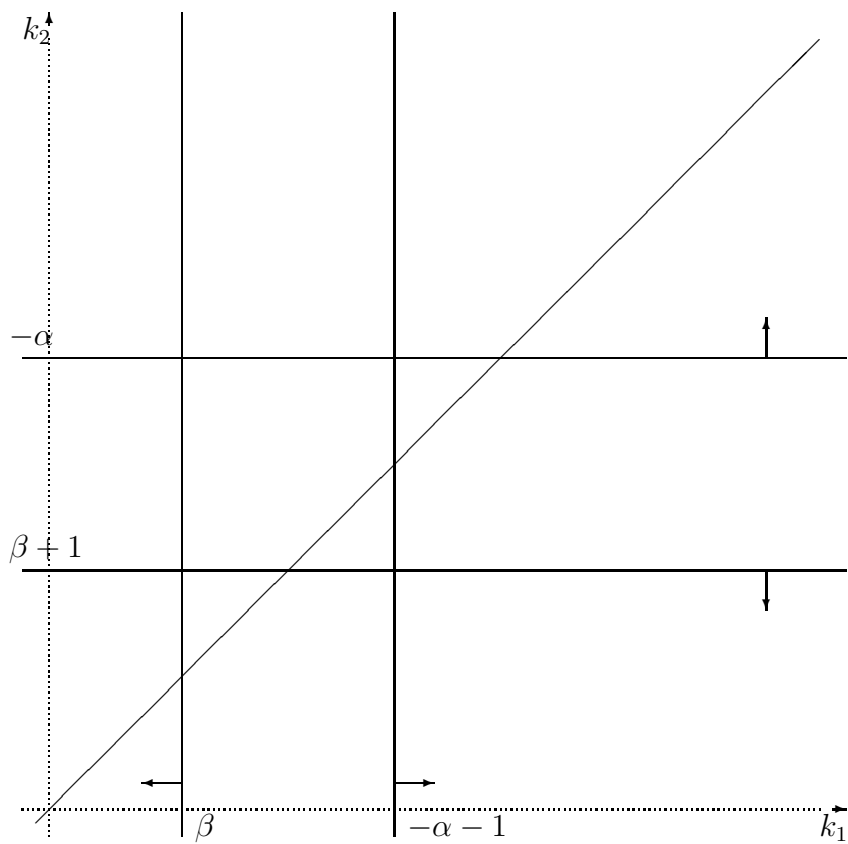


Fig.3. Structure of $\pi_{\alpha,\beta}$ with $\alpha + \beta = -2$.

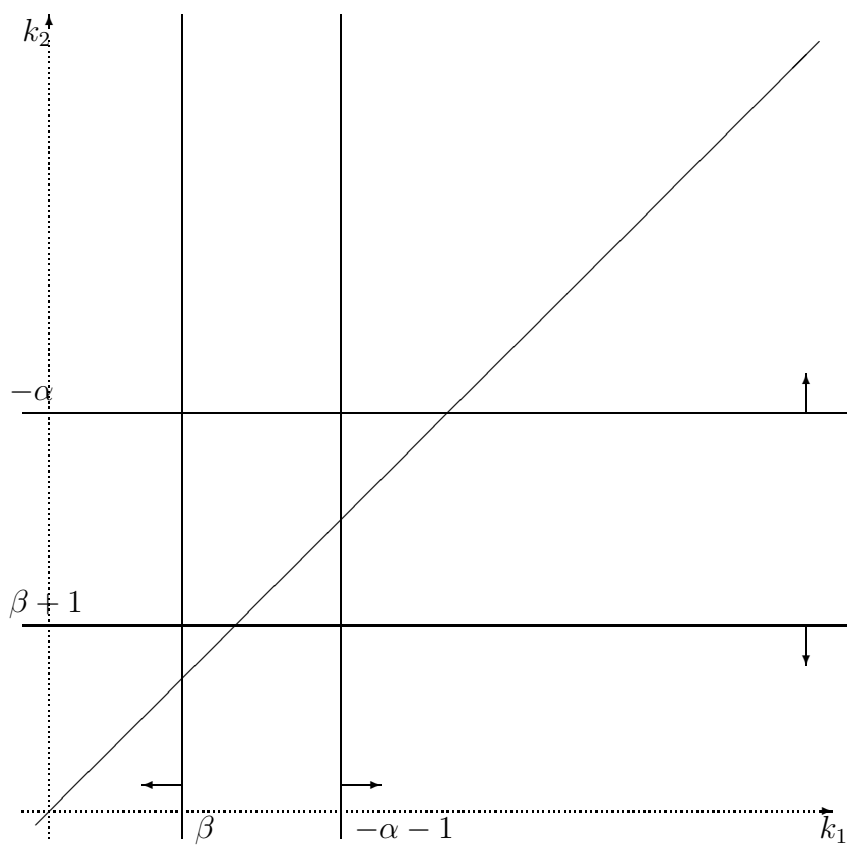


Fig.4. Structure of $\pi_{\alpha,\beta}$ with $\alpha + \beta \leq -3$.